# Decoding quantum information and complementarity principle

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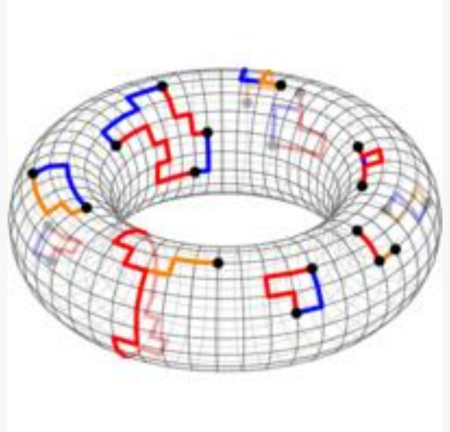
# Introduction

*Decoding QECCs*

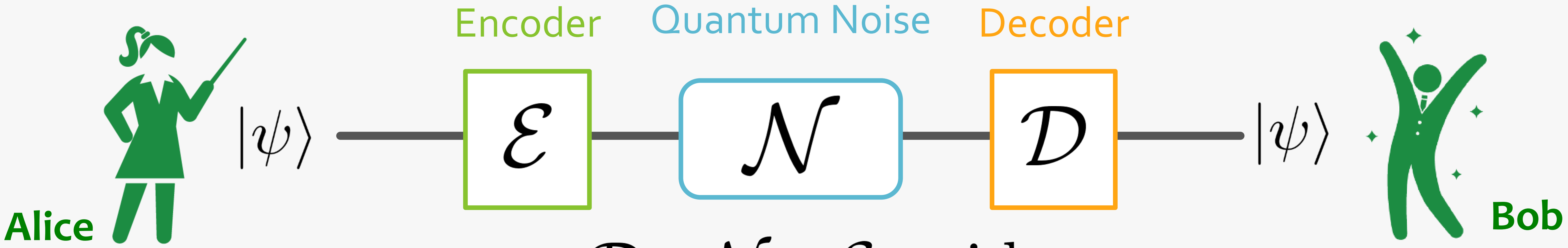
## Quantum Error Correction (QEC)

- QEC is a method to effectively cancel quantum noise by encoding and decoding.
  - Explicitly constructing a decoder is important but non-trivial.
- Decoding stabilizer codes is, in principle, straightforward.
  - Syndrome measurement → classically decode Z- and X-errors.



Decoding the toric code

**How can we decode non-stabilizer codes?**




$$\mathcal{D} \circ \mathcal{N} \circ \mathcal{E} \approx \mathrm{id}$$

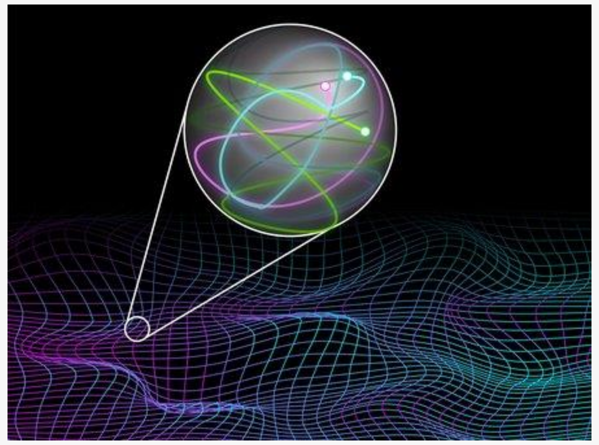
# Introduction

*Decoding QECCs*

## Why non-stabilizer codes?

**How can we decode non-stabilizer codes?**

1. In general, non-stabilizer codes achieve higher *encoding rates* than stabilizer codes.
   - E.g., the encoding rate of the toric code → $0$ as $n \to \infty$.
2. In theoretical physics, people are interested in **QEC**, **quantum chaos, and quantum gravity**.
   - Non-stabilizer codes are important.



$k$
logical qubits

Encoding rate: $r = k/n$

Alice

$|\psi\rangle$ — Encoder $\mathcal{E}$ — Quantum Noise $\mathcal{N}^{\otimes n}$ — Decoder $\mathcal{D}$ — $|\psi\rangle$

Bob

$$\mathcal{D} \circ \mathcal{N}^{\otimes n} \circ \mathcal{E} \approx \mathrm{id}$$

# Introduction

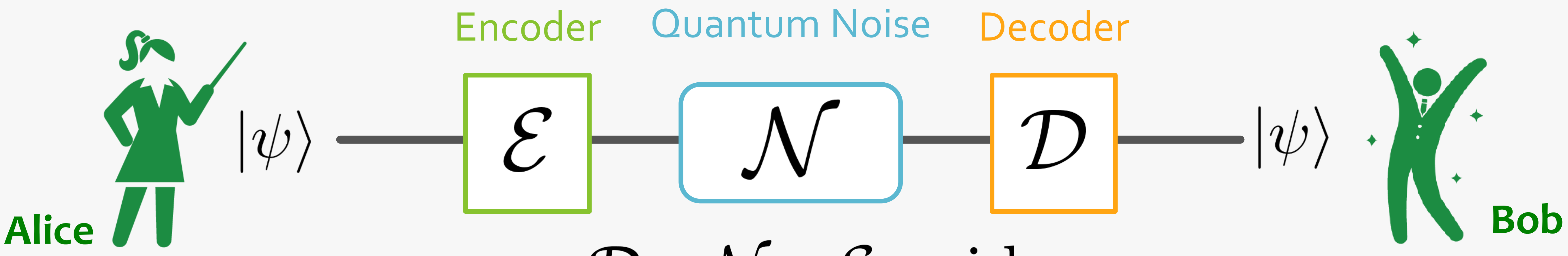
*Decoding QECCs*

## What is known about decoding non-stabilizer codes?

- Use the *Petz recovery map*. [Barnum & Knill, JMP, 2002] [Beigi, Datta, and Leditzky, JMP2016]
  - Q. algorithm for Petz is known [Gilyen et al, PRL, 2022], but it is **inefficient** and **not easy to break down**.
- An alternative approach to decoding non-stabilizer codes?

**Results today**

1. Decoding methods of stabilizer codes can be extended to general QECCs (in a certain sense).
2. In the extension, the complementarity principle comes into play.




$$\mathcal{D} \circ \mathcal{N} \circ \mathcal{E} \approx \mathrm{id}$$

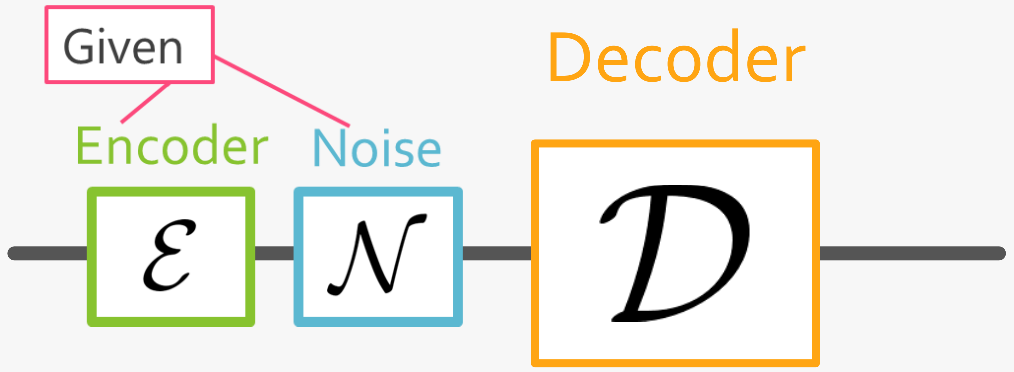
# Decoding & complementarity

*Decoding general QECCs*

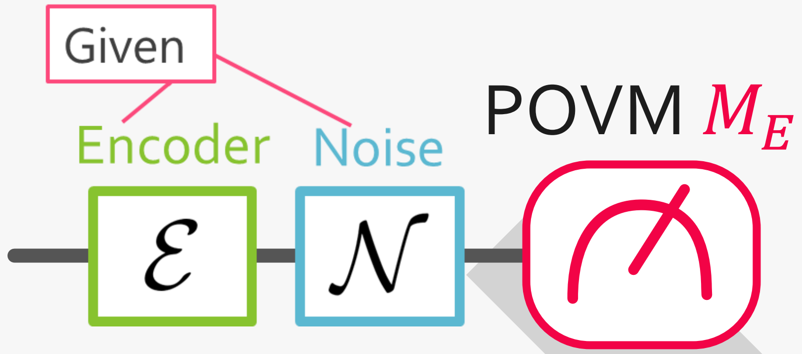
## Main result

For stabilizer codes, we infer the Z- and X-error from the **syndrome measurements**.

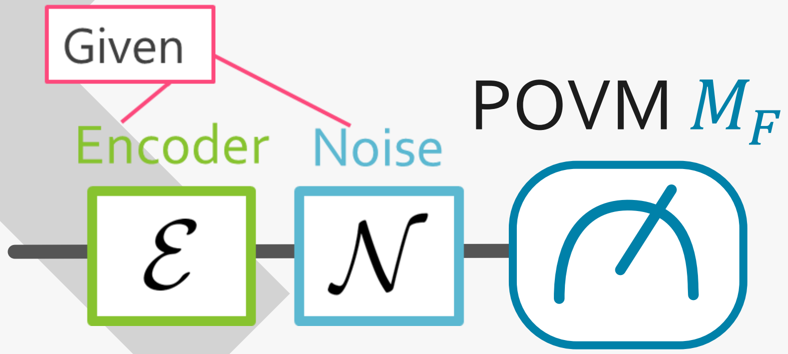
- Goal: to construct a **decoding quantum circuit** for general encodings and noisy channels.
  - Our idea is to combine **two decoding POVMs for two types of classical info**.




A decoder for
quantum information




Decoding POVM for
the $E$-basis classical info.




Decoding POVM for
the $F$-basis classical info.

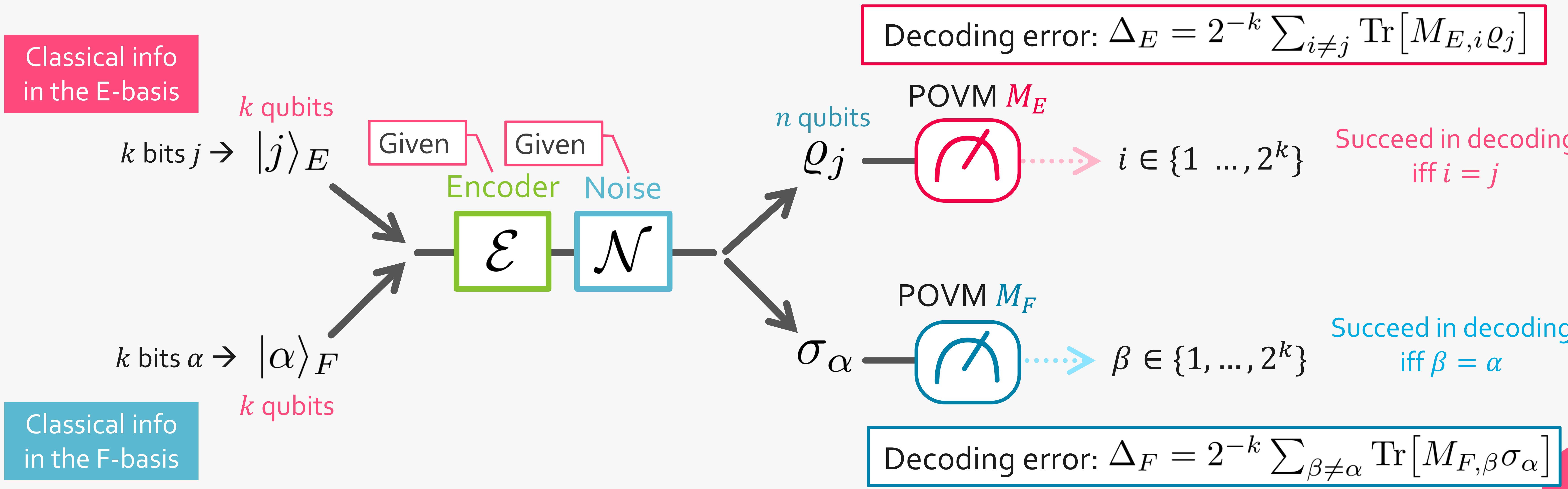
Decoding classical info?
Decoding quantum info?

# Decoding & complementarity

*Decoding general QECCs*

## Decoding classical and quantum information 1

- Decoding **classical** information (=decoding a CQ channel) by a POVM

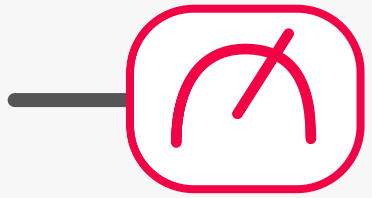


# Decoding & complementarity

*Decoding general QECCs*

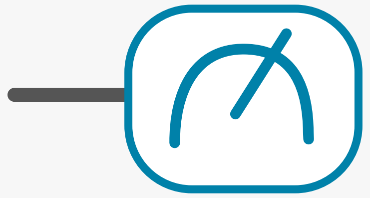
## Decoding classical and quantum information 2

- ☐ Decoding **classical** information (=decoding a CQ channel) by a POVM



POVM $M_E$

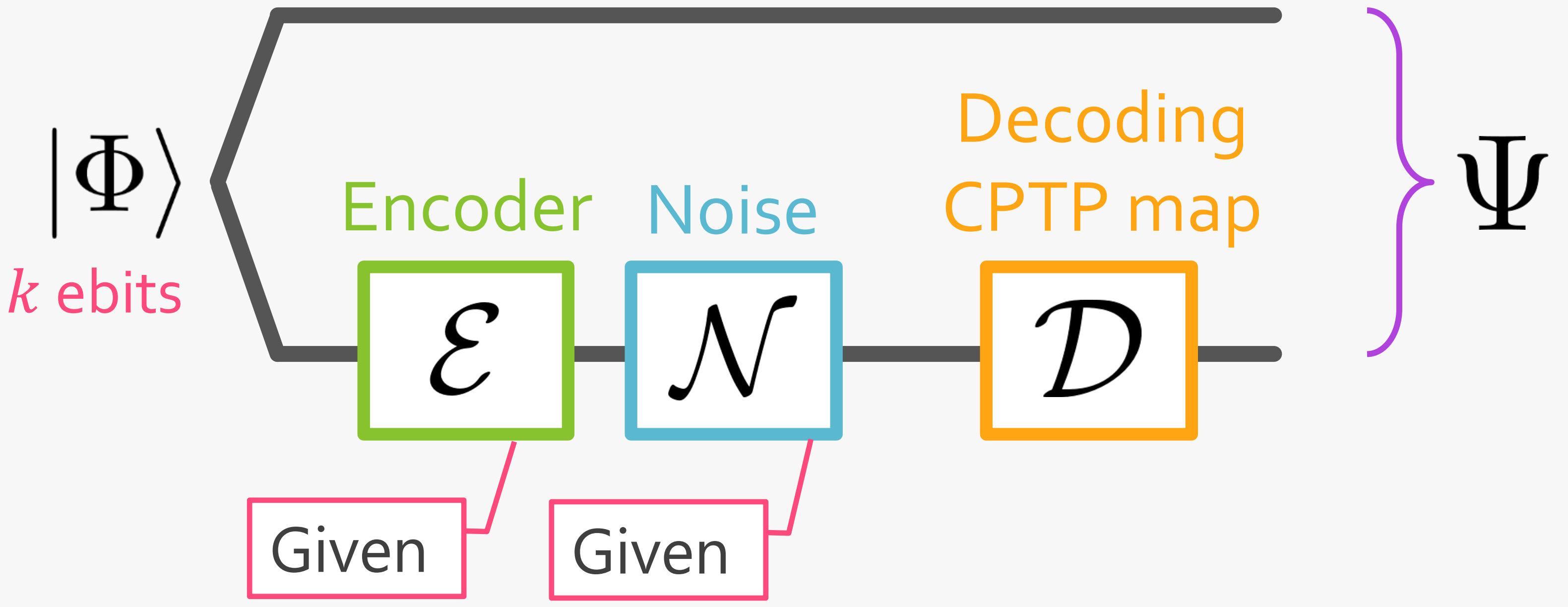
$$\Delta_E = 2^{-k} \sum_{i \neq j} \mathrm{Tr}\left[M_{E,i} \varrho_j\right]$$



POVM $M_F$

$$\Delta_F = 2^{-k} \sum_{\beta \neq \alpha} \mathrm{Tr}\left[M_{F,\beta} \sigma_\alpha\right]$$

- ☐ Decoding **quantum** information by a quantum channel




Decoding error of quantum information by the decoder $\mathcal{D}$

$$\Delta_Q = \frac{1}{2}\left\|\Psi - |\Phi\rangle\langle\Phi|\right\|_1$$

# Decoding & complementarity

*Decoding general QECCs*

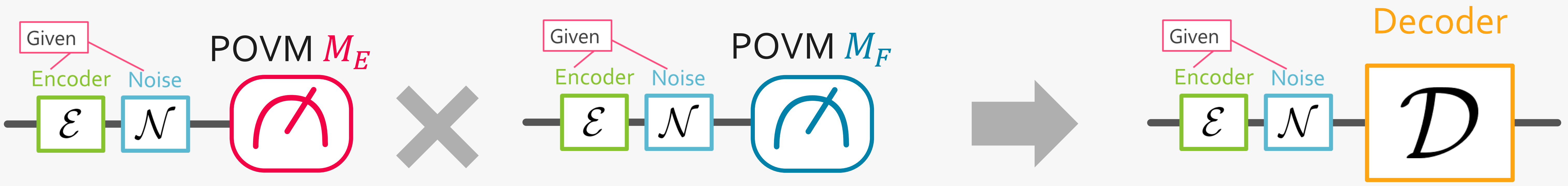
## Main result

**Theorem**

Given an encoder $\mathcal{E}$ and a noisy map $\mathcal{N}$, let $M_E$ and $M_F$ be POVMs that decode classical info in the $E$- and $F$-basis with error $\Delta_E$ and $\Delta_F$, resp. From these POVMs, a decoding circuit for quantum info can be explicitly constructed, whose decoding error $\Delta_Q$ satisfies

$$\Delta_{\mathrm{Q}} \leq \sqrt{\Delta_E(2-\Delta_E)} + \sqrt{\Delta_F} + \sqrt{\Xi(E,F)},$$

where $\Xi(E,F)$ quantifies how mutually-unbiased the bases $E$ and $F$ are.




Error $\Delta_E$ for decoding $E$-basis classical info.

Error $\Delta_F$ for decoding $F$-basis classical info.

Error $\Delta_Q$ for decoding quantum info.

# Decoding & complementarity

*Decoding general QECCs*

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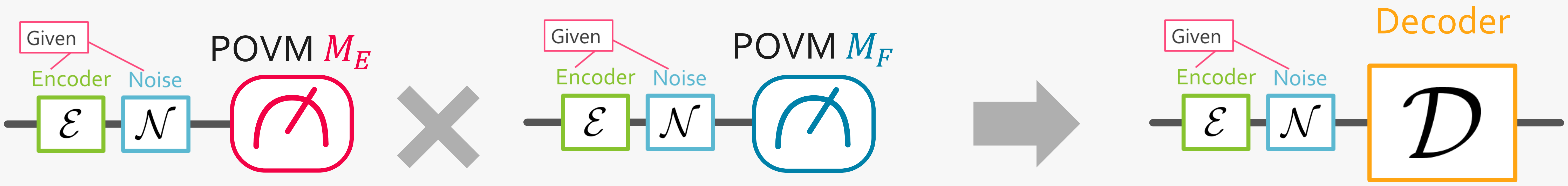


Error $\Delta_E$ for decoding $E$-basis classical info.

Error $\Delta_F$ for decoding $F$-basis classical info.

Error $\Delta_Q$ for decoding quantum info.

# Decoding & complementarity

*Decoding general QECCs*

## Mutually-unbiased bases and $\Xi(E,F)$

- A pair of two (orthonormal) bases $(E,F)$ is called *mutually-unbiased bases (MUB)* iff

$$\left|\langle \alpha_F | j_E \rangle\right| = 1/\sqrt{2^k} \quad \text{for all } \alpha, j \in \{1, \dots, 2^k\}.$$

  - One of the common characterization of "complementarity" of two bases.

- To define the quantity $\Xi(E,F)$...
  - Let $p_\alpha$ be a probability distribution $p_\alpha = \{p_{\alpha(j)} \coloneqq |\langle \alpha_F | j_E \rangle|^2\}_{j=1}^{2^k}$.
  - Classical infidelity: $\bar{F}(p_\alpha, p_\beta) = 1 - \sum_{j=1}^{2^k} \sqrt{p_{\alpha(j)} p_{\beta(j)}}$.

$$\Xi(E,F) = \max_{\alpha \in \{1,\dots,2^k\}} \bar{F}(\mathrm{unif}, p_\alpha), \quad \text{where unif is the uniform dist. over } \{1, \dots, 2^k\}.$$

A simplified definition (see the paper).

  - $\Xi(E,F) = 0$ iff $(E,F)$ is MUB & $\Xi(E,F) = 1$ iff $E = F$. → $\Xi(E,F)$ quantifies the degree of MUB.

# Decoding & complementarity

*Decoding general QECCs*

## Main result

From the two POVMs that decodes **complementary classical info**, we can explicitly construct a **decoding circuit** for **quantum** info.

**Theorem**
Given an encoder $\mathcal{E}$ and a noisy map $\mathcal{N}$, let $M_E$ and $M_F$ be POVMs that decode classical info in the $E$- and $F$-basis with error $\Delta_E$ and $\Delta_F$, resp. From these POVMs, a decoding circuit for quantum info can be explicitly constructed, whose decoding error $\Delta_Q$ satisfies

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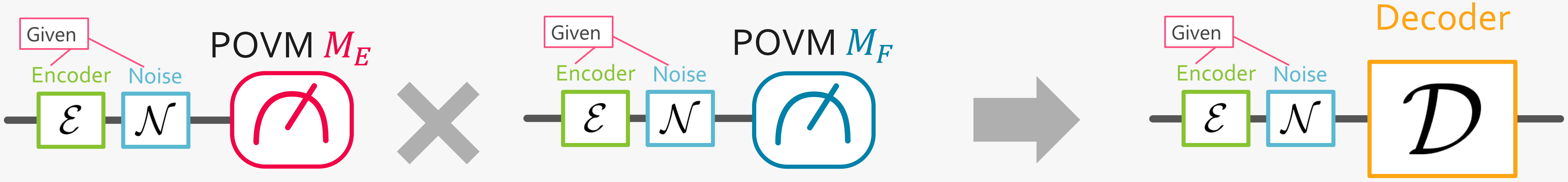



Error $\Delta_E$ for decoding $E$-basis classical info.

Error $\Delta_F$ for decoding $F$-basis classical info.

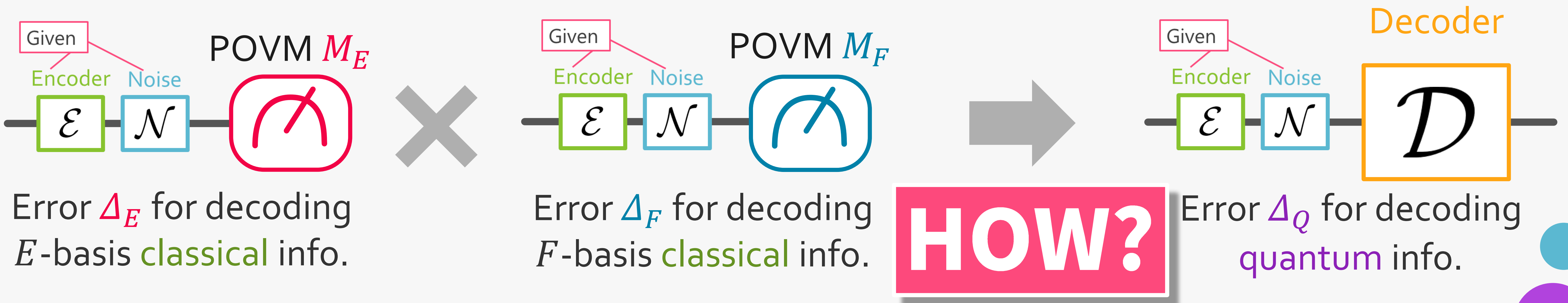
Error $\Delta_Q$ for decoding quantum info.

# Decoding & complementarity

*Decoding general QECCs*

## Implications of our result

- **Natural that complementarity comes into play in decoding quantum information.**
  - N. Bohr: "to characterize a quantum system, we need complementary information, such as position and momentum".
  - Our result shows the significance of the **complementary feature in QEC**.
- **Our final goal: to construct a decoding circuit for general QECCs**
  - Our result makes a good step: it breaks down a quantum decoder to POVMs for classical info.



1 Introduction

*Decoding QECCs*

2 Summary of the results

*Decoding general QECCs*

## 3 Decoding circuits for general QECCs

*From CSS codes to general QECCs*

4 Conclusion

# Decoding circuits for general QECCs

*Decoding general QECCs*

## Extending the decoder of CSS codes to general QECCs 1

<u>Our idea</u>

To extend the decoder of CSS codes to general QECCs

- CSS codes are stabilizer codes made of <u>two classical linear codes</u> $C_1$ and $C_2$.
  - $C_1/C_2$ defines the logical-Z basis, and $C_2^\perp/C_1^\perp$ the logical-X basis.

$$|\bar{j}\rangle_Z \propto \sum_{\omega \in C_2} |j+\omega\rangle \qquad |\bar{\alpha}\rangle_X \propto \sum_{v \in C_1^\perp} |\alpha+v\rangle$$

- Decoding CSS codes.

1. **<u>Syndrome measurements</u>** → error syndromes ∈ $\{\pm 1\}^{n-k}$.
2. Decode the Z info by the decoder $f_1$ of $C_1/C_2$ and the X info by $f_2$ of $C_2^\perp/C_1^\perp$.

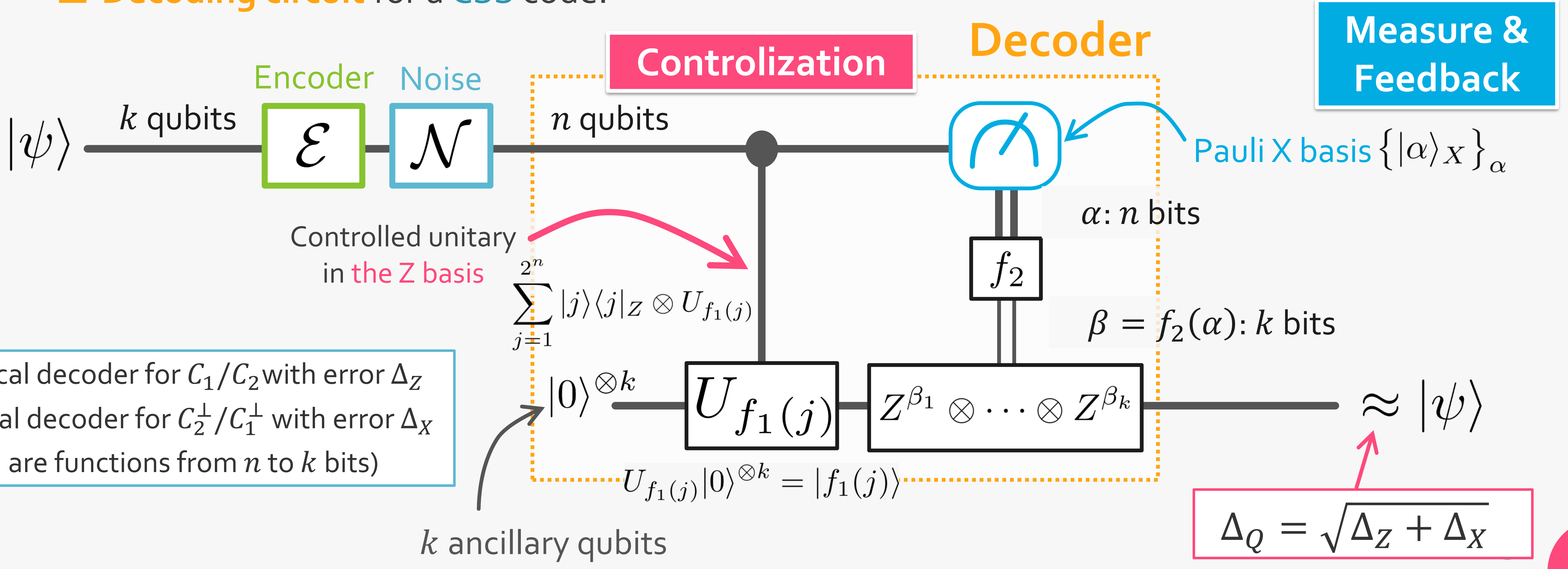
➡ Doesn't seem to be extendable to non-stabilizer codes…., but possible!

# Decoding circuits for general QECCs



## Extending the decoder of CSS codes to general QECCs 2

- Decoding circuit for a CSS code.




$f_1$: classical decoder for $C_1/C_2$ with error $\Delta_Z$
$f_2$: classical decoder for $C_2^\perp/C_1^\perp$ with error $\Delta_X$
(both are functions from $n$ to $k$ bits)

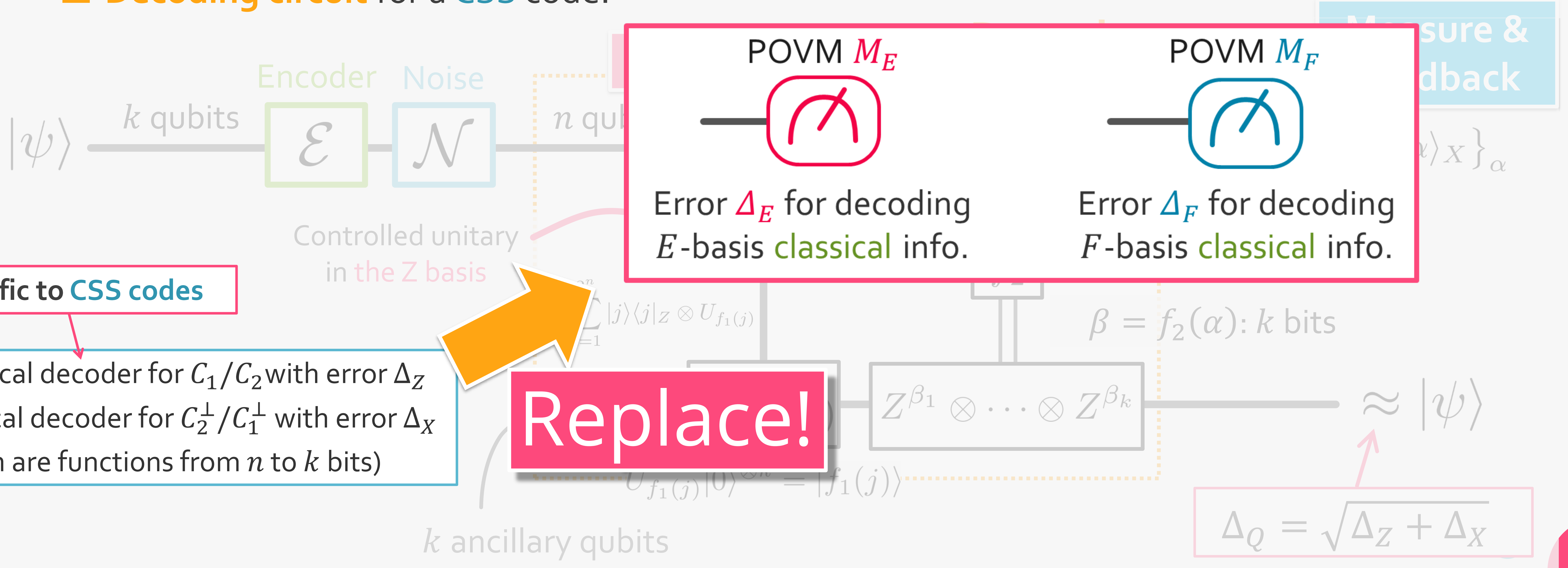
$\Delta_Q = \sqrt{\Delta_Z + \Delta_X}$

# Decoding circuits for general QECCs

*Decoding general QECCs*

## Extending the decoder of CSS codes to general QECCs 2

- **Decoding circuit** for a CSS code.




**Specific to CSS codes**

$f_1$: classical decoder for $C_1/C_2$ with error $\Delta_Z$
$f_2$: classical decoder for $C_2^\perp/C_1^\perp$ with error $\Delta_X$
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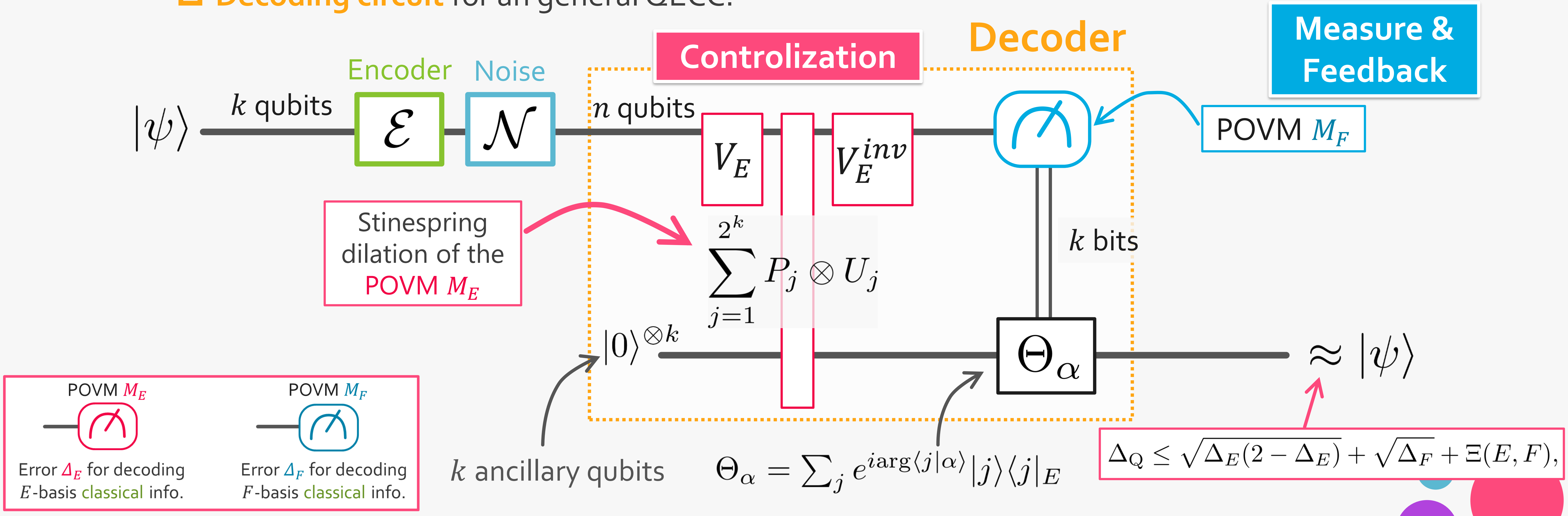
# Decoding circuits for general QECCs

*Decoding general QECCs*

## Extending the decoder of CSS codes to general QECCs 3

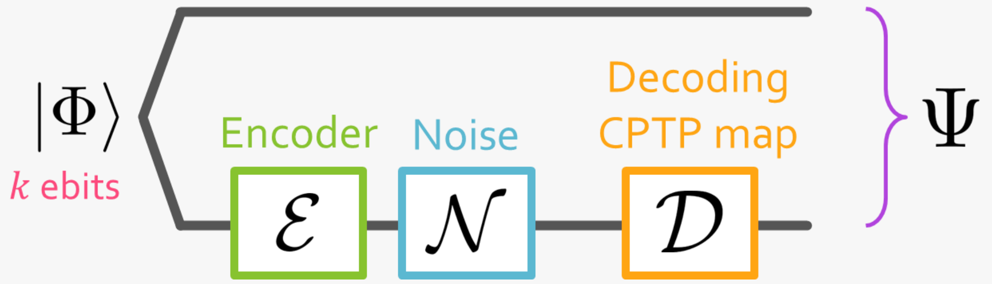
- **Decoding circuit** for an general QECC.



# Decoding circuits for general QECCs

*Decoding general QECCs*

## Decoding error analysis 1

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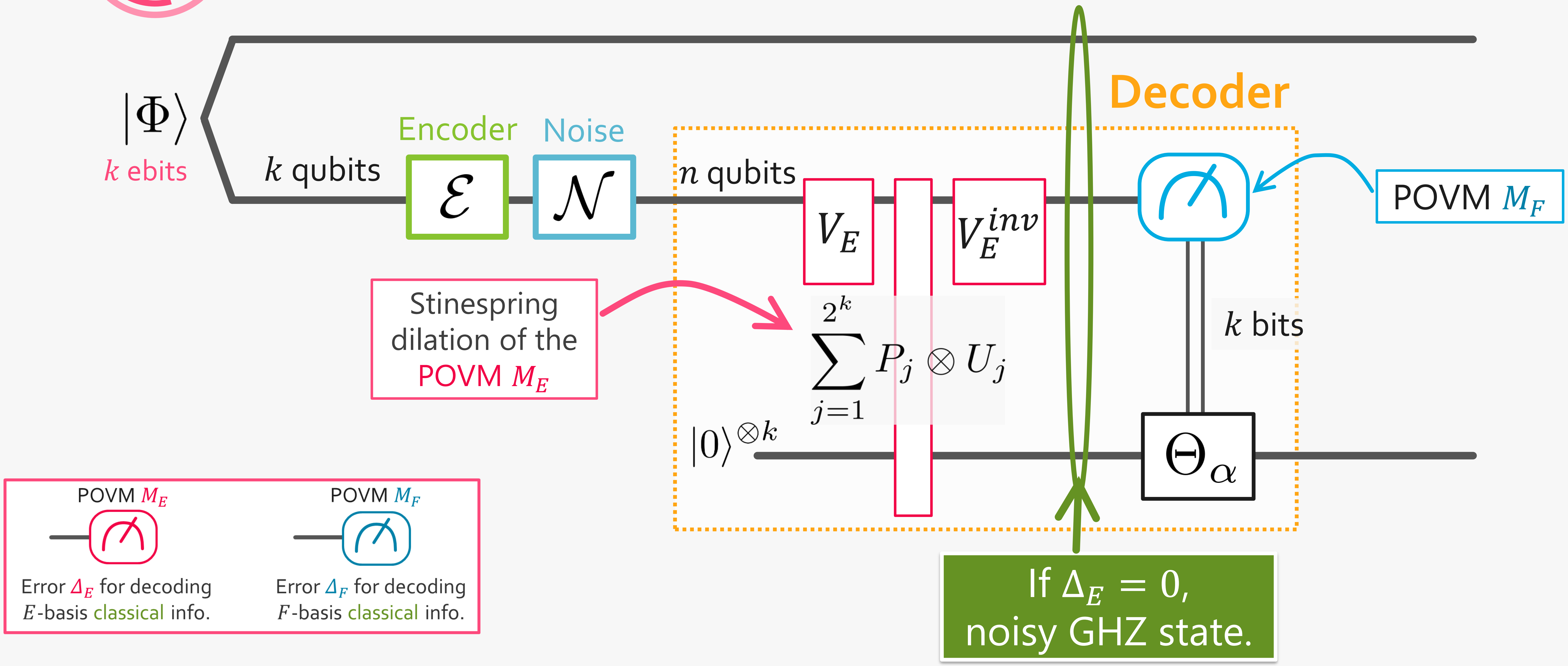
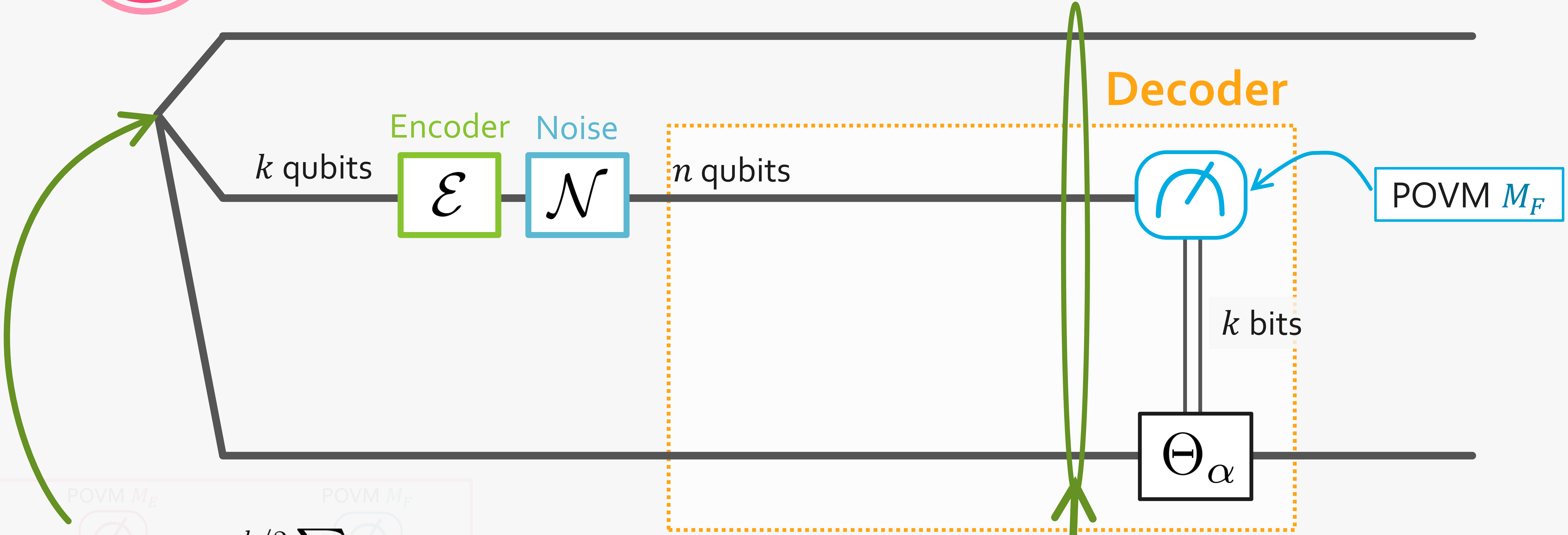


# Decoding circuits for general QECCs

*Decoding general QECCs*

## Decoding error analysis 1

**Goal: to transform a noisy GHZ to MES!**




$$|\mathrm{GHZ}\rangle_E = 2^{-k/2} \sum_j |j\rangle_E \otimes |j\rangle_E \otimes |j\rangle_E$$

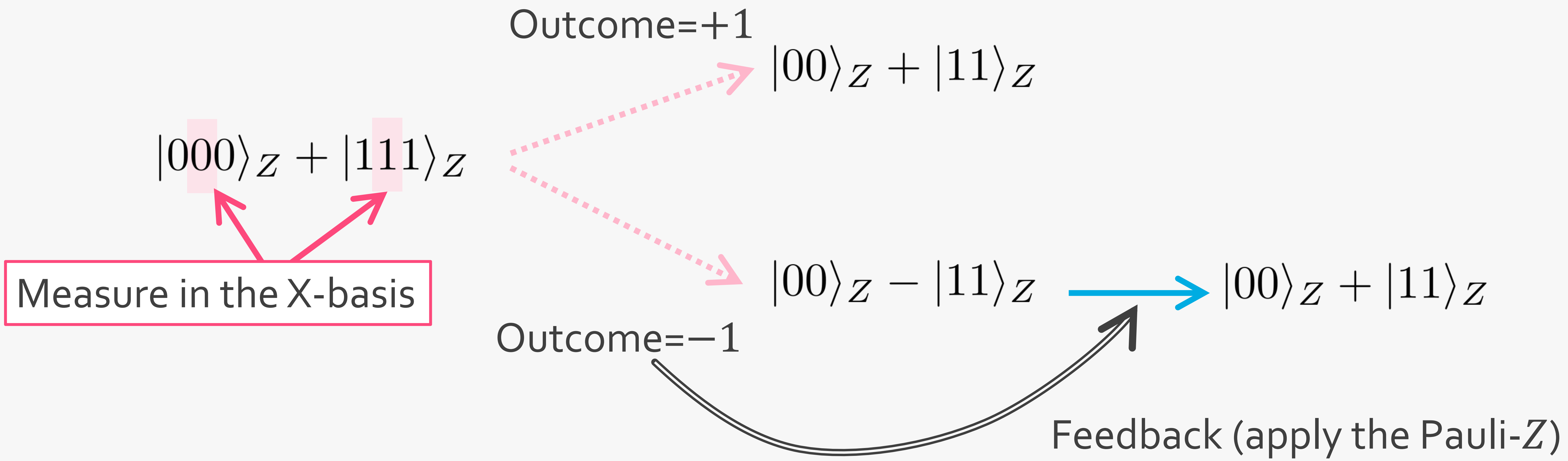
If $\Delta_E = 0$,
noisy GHZ state.

# Decoding circuits for general QECCs

*Decoding general QECCs*

## Decoding error analysis 2


- A GHZ state can be transformed to the MES by measurement and feedback.




**If the two bases are NOT MUB, this transformation is approximate.**

# Decoding circuits for general QECCs

*Decoding general QECCs*

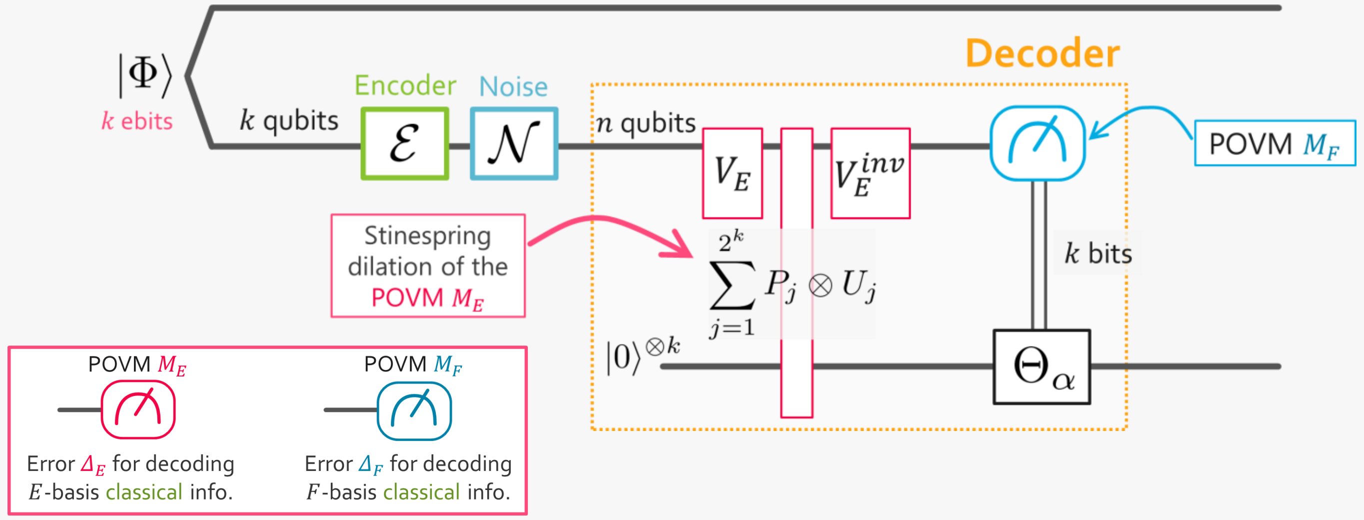
## Decoding error analysis: end

- Decoding error = (State is approx. a noisy GHZ) + (Error on transforming the noisy GHZ → MES)

$$\Delta_{\mathrm{Q}} \leq \underbrace{\sqrt{\Delta_E(2-\Delta_E)}}_{\text{Approximate the noisy state by a noisy GHZ}} + \underbrace{\sqrt{\Delta_F} + \Xi(E,F)}_{\text{Error on transforming the noisy GHZ to MES}}$$

Approximate the noisy state by a noisy GHZ

Error on transforming the noisy GHZ to MES

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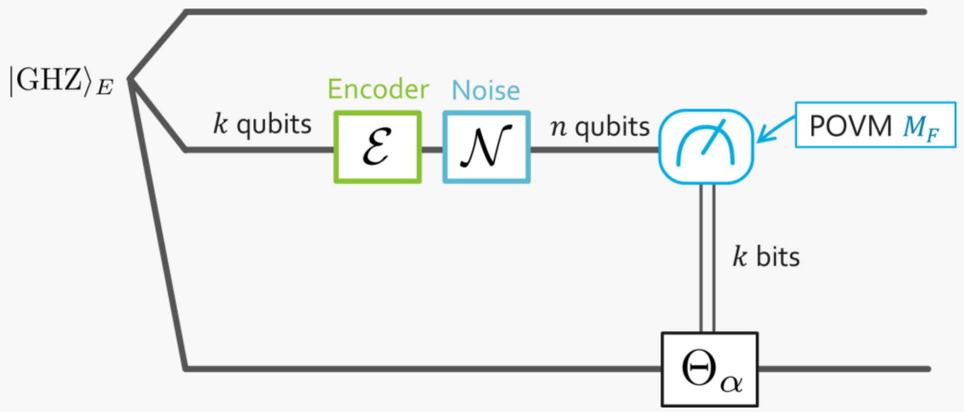


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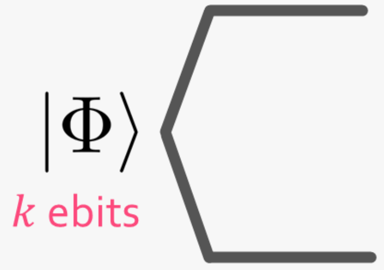


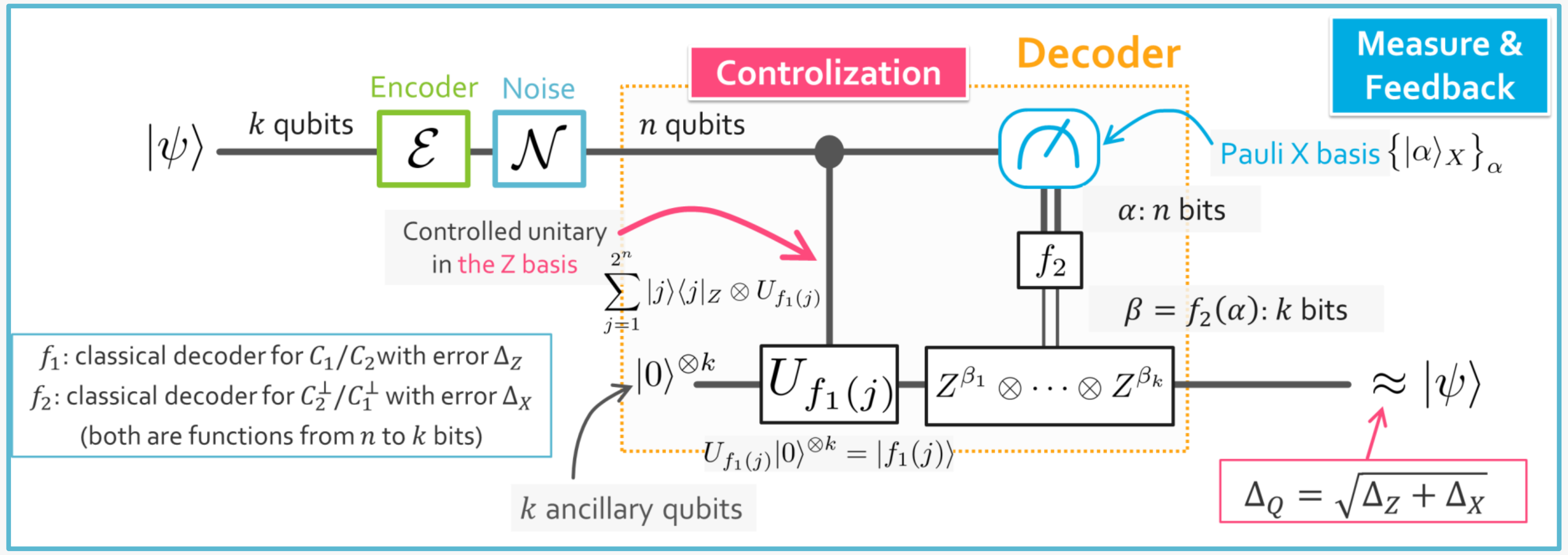
# Decoding circuits for general QECCs

*Decoding general QECCs*

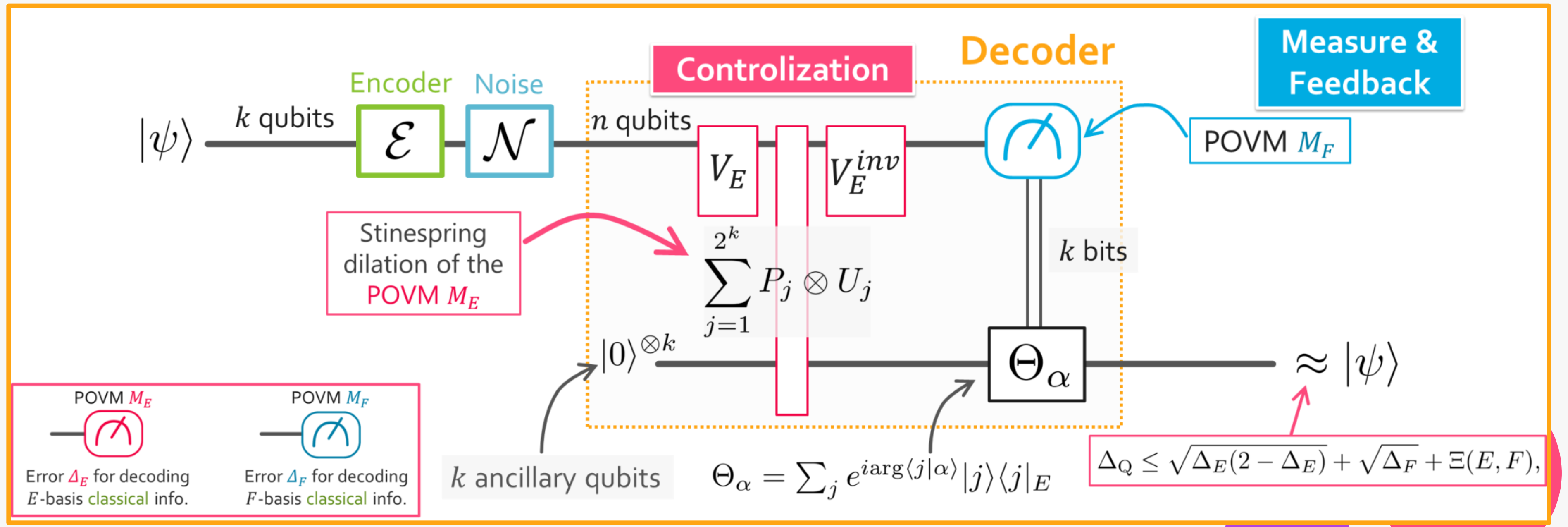
## Decoding circuits –similarity and difference-

- **Decoding circuits for CSS codes and non-stabilizer codes**
  - In both cases, controlize one measurement, and use the other for measurement & feedback.
  - Decoding error is $\Delta_Q = \sqrt{\Delta_Z + \Delta_X}$ for CSS, and $\Delta_Q \leq \sqrt{\Delta_E(2-\Delta_E)} + \sqrt{\Delta_F}$ for non-stabilizer (when $(E, F)$ is MUB).
  - This is due to the **back-action of the measurement**. No back-action in CSS, but it exists in general.

Decoding circuit for CSS codes

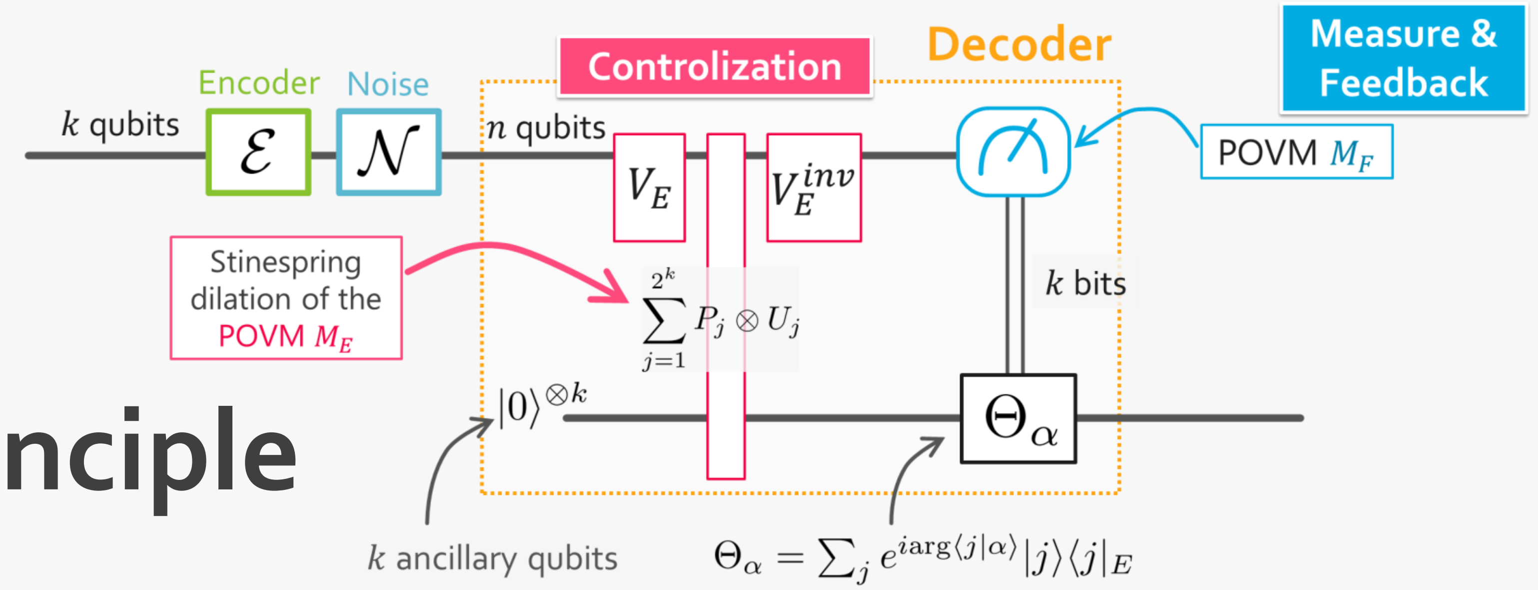



Decoding circuit for general QECCs



# Conclusion

## Decoding and complementarity principle

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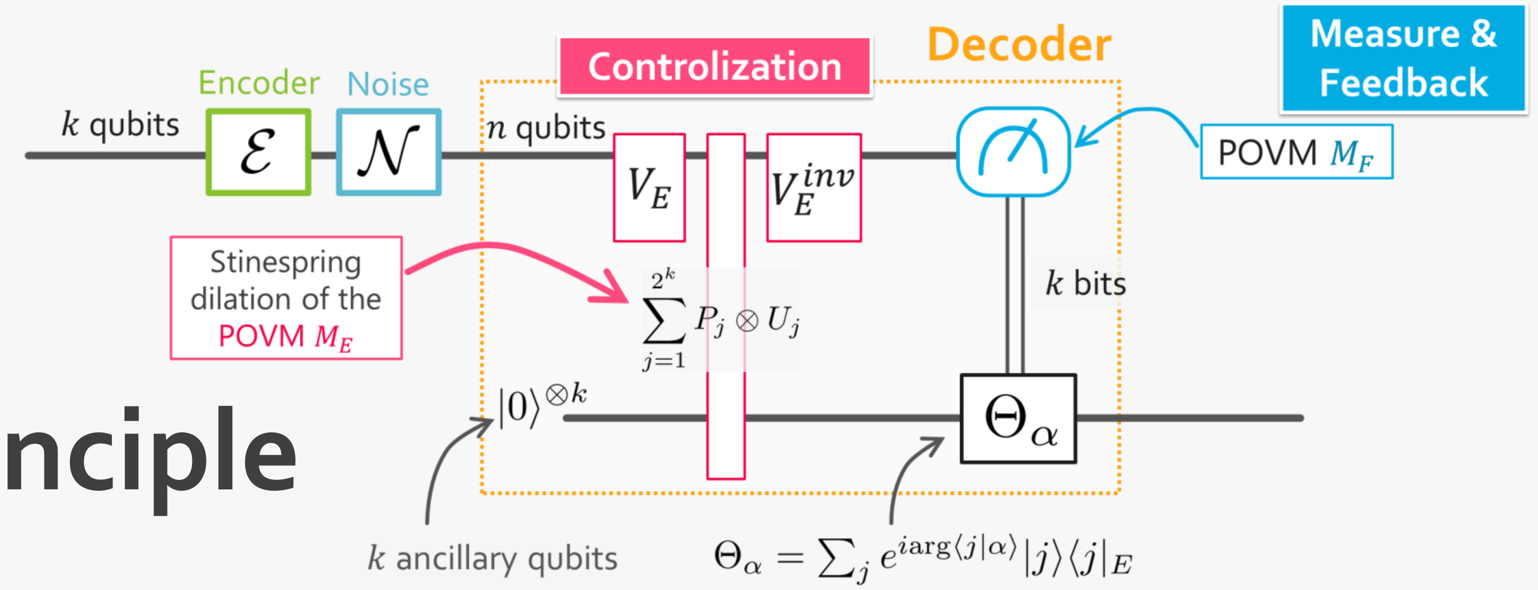
where $\Xi(E,F)$ quantifies how mutually-unbiased the bases $E$ and $F$ are ($\Xi(E,F) = 0 \Leftrightarrow$(E,F) is MUB).

- **From the two POVMs that decodes classical info defined in the MUB, we can explicitly construct a decoding circuit for quantum info.**
  - Complementarity (in the sense of MUB) is important in decoding quantum information.

# Conclusion




## Decoding and complementarity principle

$$\Delta_Q \leq \sqrt{\Delta_E(2-\Delta_E)} + \sqrt{\Delta_F} + \Xi(E,F),$$

1. **How can we find good POVMs for classical info?**
   - Our result is just to convert **POVMs** to a **quantum decoder**.
   - A PGM should work, but it's explicit construction by quantum circuits is not straightforward.

→ Our recent result: a decoder can be explicitly constructed from scratch (TBA).

2. **The bound is probably NOT tight.**
   - We provided one construction of a decoder $\mathcal{D}$. A better construction may exist.
   - In certain cases, $\Delta_Q = 0$ when $\Delta_E, \Delta_F = 0$ <u>even if $(E, F)$ is NOT MUB</u>.

→ MUBness should appear in the bound in a different form.

# Thank you

• • •

for your attention

arXiv:2210.06661

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