Barycentric Rényi divergences

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MOMENTUM OF INNOVATION

Magyar Tudományos Akadémia 1825

Lendület program

Single-shot state discrimination

Communication problem: Alice wants to send one bit of information to Bob, encoded into a quantum system.

Alice		Bob		
$0 \mapsto \varrho$		POVM		
	$\longrightarrow$		$\longrightarrow$	decoded message $= 0$ or 1.
$1 \mapsto \sigma$		$T_0 = T, \quad T_1 = I - T$		

Error probabilities:

type I: $\alpha(T) := \mathrm{Prob}(\text{decoded} = 1 | \text{message} = 0) = \mathrm{Tr}\, \varrho(I - T)$

type II: $\beta(T) := \mathrm{Prob}(\text{decoded} = 0 | \text{message} = 1) = \mathrm{Tr}\, \sigma T.$

Type I can be made 0 by choosing $T := I$, but then type II $= 1$.

Vice versa, type II $= 0$ with $T = 0$, but then type I $= 1$.

Trade-off between the two error probabilities.

$$\exists T : \quad \alpha(T) = 0 = \beta(T) \qquad \Longleftrightarrow \qquad \mathrm{supp}\, \varrho \perp \mathrm{supp}\, \sigma$$

Asymptotic state discrimination

The error can be reduced by introducing redundancy.

Alice | Bob

$$0 \mapsto \underbrace{00\ldots0}_{n \text{ times}} \mapsto \varrho^{\otimes n}$$

$$1 \mapsto \underbrace{11\ldots1}_{n \text{ times}} \mapsto \sigma^{\otimes n}$$

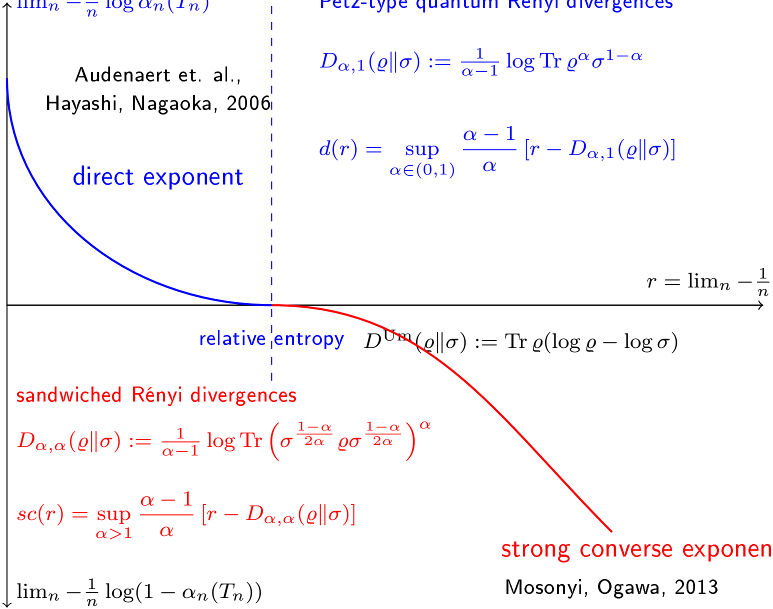
$\longrightarrow$ POVM $T_0 = T, \quad T_1 = I - T$ $\longrightarrow$ decoded message $= 0$ or 1.

Error probabilities:

$$\alpha_n(T) := \operatorname{Tr} \varrho^{\otimes n}(I - T), \qquad \beta_n(T) := \operatorname{Tr} \sigma^{\otimes n} T$$

What is the best asymptotics along arbitrary test sequences $0 \le T_n \le I_{\mathcal{H}^{\otimes n}}$, $n \in \mathbb{N}$?

Trade-off relations for binary state discrimination



Petz-type quantum Rényi divergences

$$D_{\alpha,1}(\varrho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho^{\alpha}\sigma^{1-\alpha}$$

$$d(r) = \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left[r - D_{\alpha,1}(\varrho\|\sigma)\right]$$

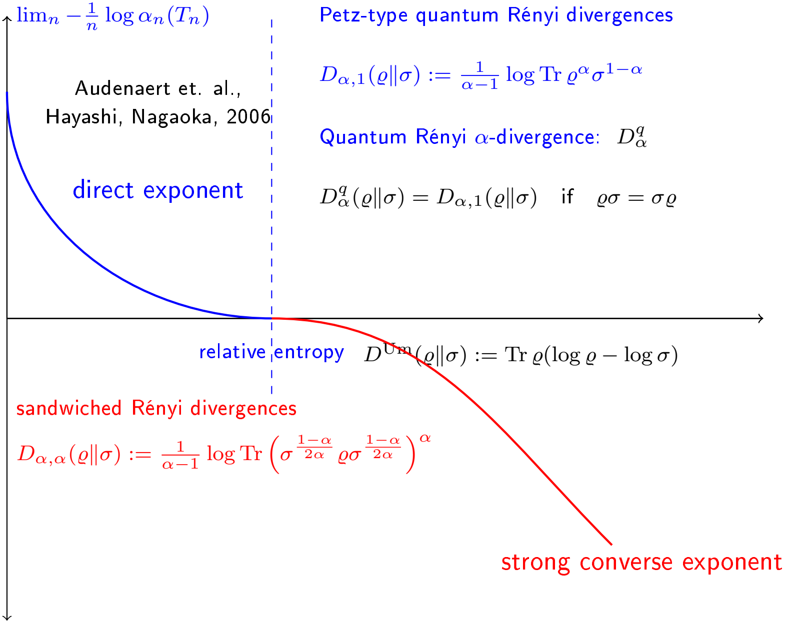
$$D^{\mathrm{Um}}(\varrho\|\sigma) := \operatorname{Tr}\varrho(\log\varrho - \log\sigma)$$

sandwiched Rényi divergences

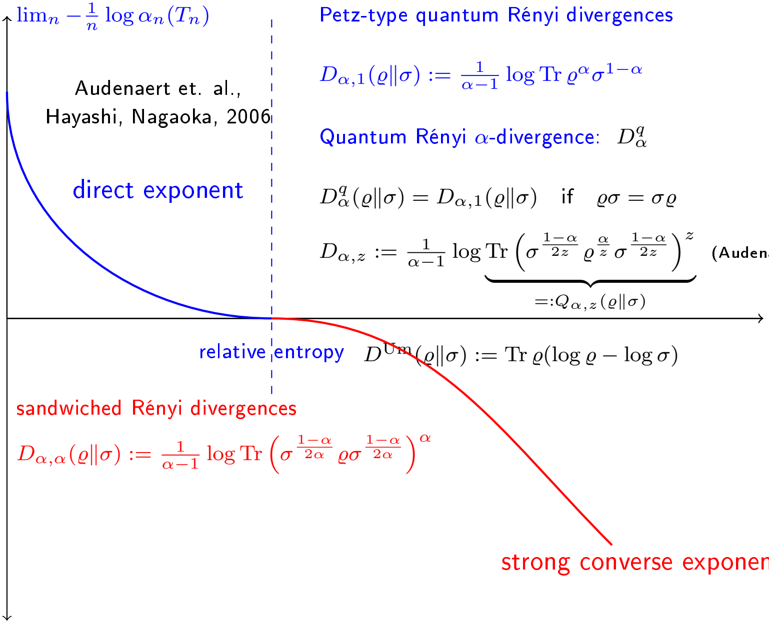
$$D_{\alpha,\alpha}(\varrho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\varrho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}$$

$$sc(r) = \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left[r - D_{\alpha,\alpha}(\varrho\|\sigma)\right]$$

Trade-off relations for binary state discrimination



Trade-off relations for binary state discrimination



Audenaert et. al., Hayashi, Nagaoka, 2006

Petz-type quantum Rényi divergences

$$D_{\alpha,1}(\varrho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr} \varrho^{\alpha}\sigma^{1-\alpha}$$

Quantum Rényi α-divergence: D_α^q

$$D_\alpha^q(\varrho\|\sigma) = D_{\alpha,1}(\varrho\|\sigma) \quad \text{if} \quad \varrho\sigma = \sigma\varrho$$

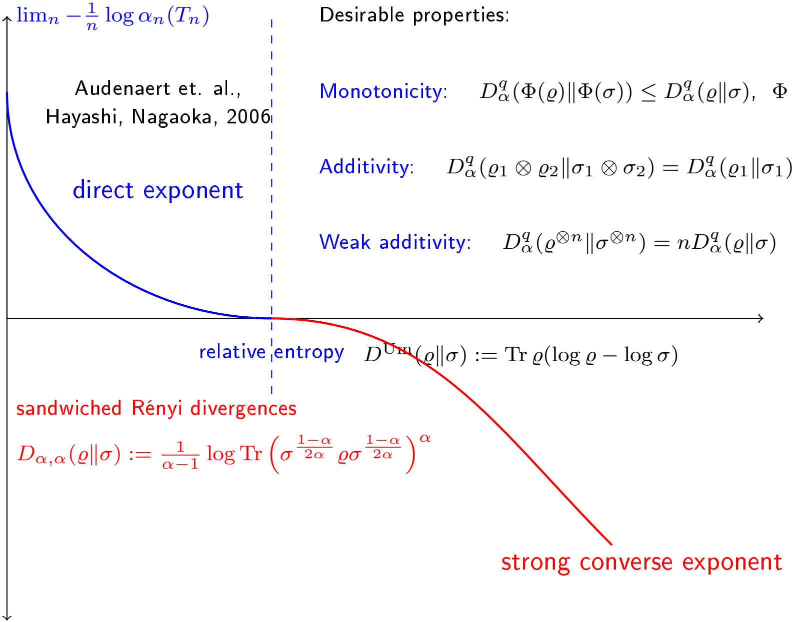
$$D_{\alpha,z} := \frac{1}{\alpha-1}\log \underbrace{\operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2z}}\varrho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right)^z}_{=:Q_{\alpha,z}(\varrho\|\sigma)}$$ (Audenaert, Datta 2013)

relative entropy $D^{\mathrm{Um}}(\varrho\|\sigma) := \operatorname{Tr}\varrho(\log\varrho - \log\sigma)$

sandwiched Rényi divergences

$$D_{\alpha,\alpha}(\varrho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\varrho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}$$

Trade-off relations for binary state discrimination



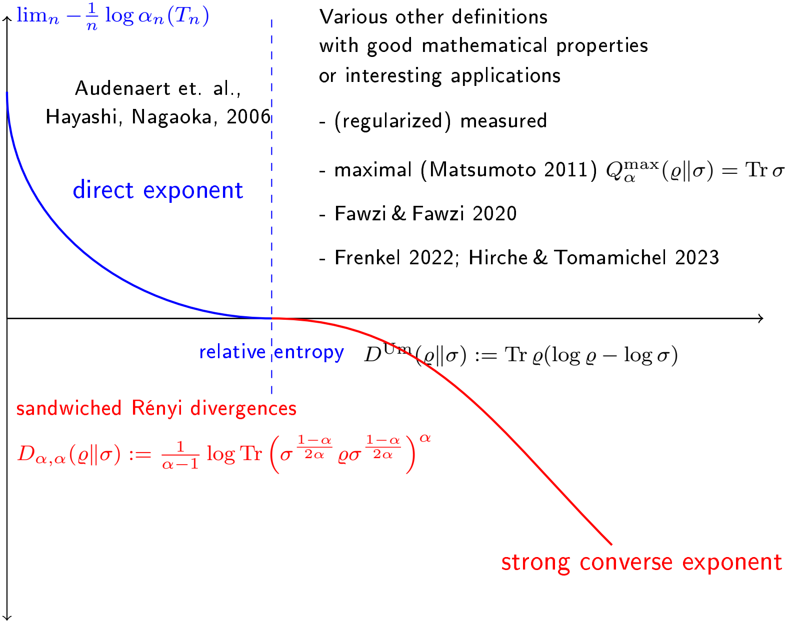
Desirable properties:

Monotonicity: $D_\alpha^q(\Phi(\varrho)\|\Phi(\sigma)) \le D_\alpha^q(\varrho\|\sigma)$, Φ CPTP

Additivity: $D_\alpha^q(\varrho_1\otimes\varrho_2\|\sigma_1\otimes\sigma_2) = D_\alpha^q(\varrho_1\|\sigma_1) + D_\alpha^q(\varrho_2\|\sigma_2)$

Weak additivity: $D_\alpha^q(\varrho^{\otimes n}\|\sigma^{\otimes n}) = nD_\alpha^q(\varrho\|\sigma)$

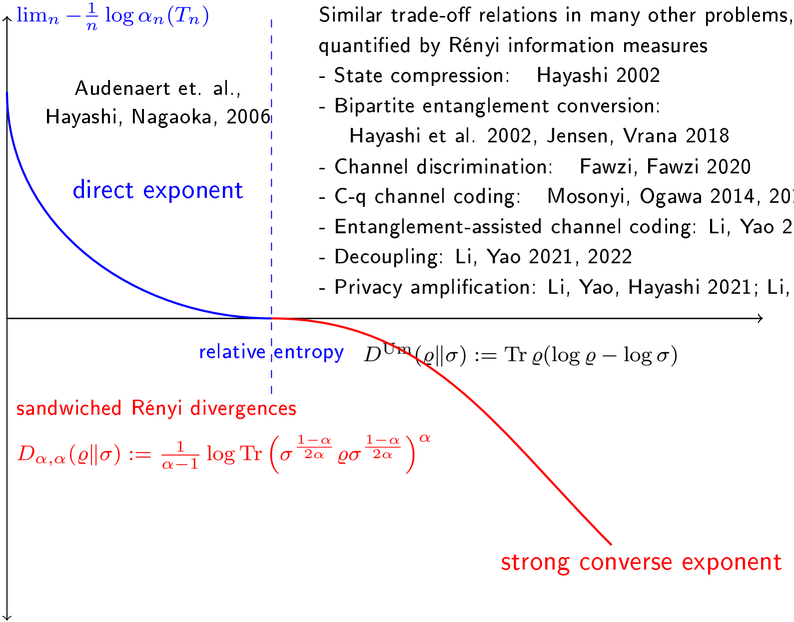
Trade-off relations for binary state discrimination



Various other definitions
with good mathematical properties
or interesting applications

- (regularized) measured
- maximal (Matsumoto 2011) $Q_\alpha^{\max}(\varrho\|\sigma) = \operatorname{Tr}\sigma(\sigma^{-\frac{1}{2}}\varrho\sigma^{-\frac{1}{2}})^{\alpha}$
- Fawzi & Fawzi 2020
- Frenkel 2022; Hirche & Tomamichel 2023

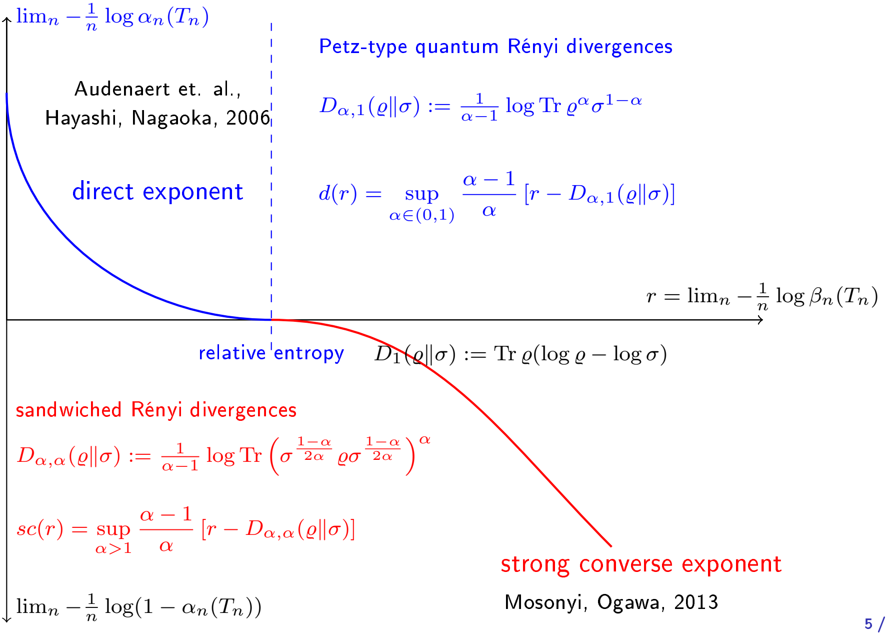
Trade-off relations for binary state discrimination

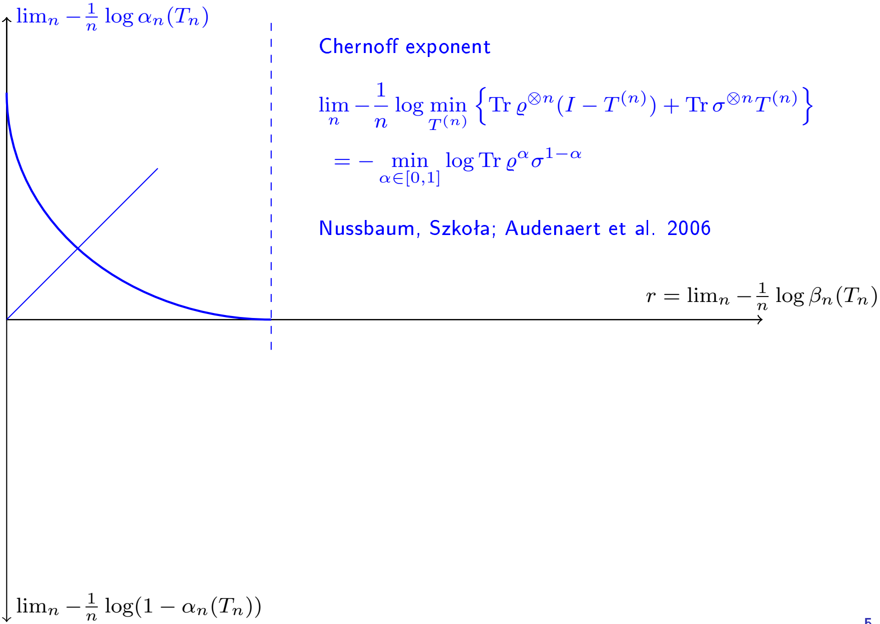


Similar trade-off relations in many other problems, quantified by Rényi information measures

- State compression: Hayashi 2002
- Bipartite entanglement conversion:
  Hayashi et al. 2002, Jensen, Vrana 2018
- Channel discrimination: Fawzi, Fawzi 2020
- C-q channel coding: Mosonyi, Ogawa 2014, 2018
- Entanglement-assisted channel coding: Li, Yao 2022
- Decoupling: Li, Yao 2021, 2022
- Privacy amplification: Li, Yao, Hayashi 2021; Li, Yao, 2022

Symmetric state discrimination



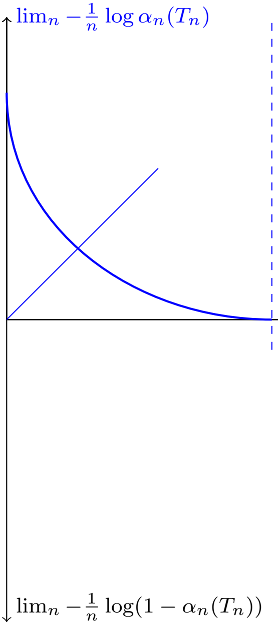


Chernoff exponent

$$\lim_n -\frac{1}{n}\log\min_{T^{(n)}}\left\{\operatorname{Tr}\varrho^{\otimes n}(I-T^{(n)})+\operatorname{Tr}\sigma^{\otimes n}T^{(n)}\right\}$$
$$=-\min_{\alpha\in[0,1]}\log\operatorname{Tr}\varrho^{\alpha}\sigma^{1-\alpha}$$

Nussbaum, Szkoła; Audenaert et al. 2006

Symmetric state discrimination



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$$= -\min_{\alpha\in[0,1]} \log \mathrm{Tr}\,\varrho^{\alpha}\sigma^{1-\alpha}$$

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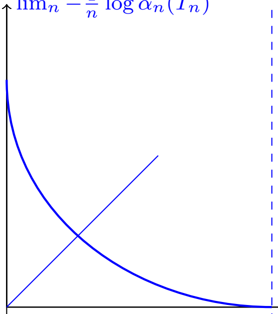
State exclusion

$$\lim_n -\frac{1}{n}\log \min_{T_1^{(n)},\ldots,T_r^{(n)}} \sum_{i=1}^{r} \mathrm{Tr}\,\varrho_i^{\otimes n} T_i^{(n)}$$

$$= -\min_{\alpha_1+\ldots+\alpha_r=1} \log \mathrm{Tr}\,\varrho_1^{\alpha_1}\ldots\varrho_r^{\alpha_r}$$

if $\varrho_i\varrho_j = \varrho_j\varrho_i$ Nussbaum, Mishra, Wilde 2023

Symmetric state discrimination



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What is the right definition of multivariate quantum Rényi divergences?

State exclusion

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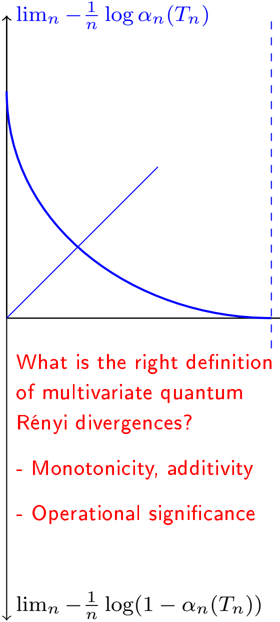

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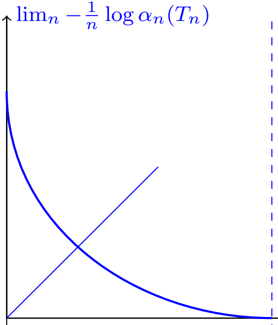
What is the right definition of multivariate quantum Rényi divergences?

- Monotonicity, additivity

- Operational significance

Symmetric state discrimination



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What is the right definition of multivariate quantum Rényi divergences?

- Monotonicity, additivity

- Operational significance

(Furuya, Lashkari, Ouseph 2021)

(Bunth, Vrana 2021)

State conversion

$\varrho_1, \ldots, \varrho_r, \quad \varrho'_1, \ldots, \varrho'_r$ given

Single-shot: $\exists\ \Phi$ CPTP: $\quad \Phi(\varrho_i) = \varrho'_i,\ i \in [r]\ \ ?$

Necessary: $\quad D_P^q$ r-variable monotone quantum Rényi-divergence

$$D_P^q((\varrho_i)_{i\in[r]}) \geq D_P^q((\varrho'_i)_{i\in[r]})$$

Sufficient conditions in terms of conditional min-entropy
(Gour, Jennings, Buscemi, Duan, Marvian 2018)

State conversion

$\varrho_1, \dots, \varrho_r, \quad \varrho'_1, \dots, \varrho'_r$ given

Single-shot: $\exists\ \Phi$ CPTP: $\quad \Phi(\varrho_i) = \varrho'_i,\ i \in [r]$?

Multi-copy: $\exists\ n \in \mathbb{N}$, Φ CPTP: $\quad \Phi(\varrho_i^{\otimes n}) = \varrho_i'^{\otimes n},\ i \in [r]$?

Necessary: $\quad D_P^q$ r-variable monotone, weakly additive quantum Rényi-divergence

$$D_P^q((\varrho_i)_{i\in[r]}) \geq D_P^q((\varrho'_i)_{i\in[r]})$$

State conversion

$\varrho_1, \dots, \varrho_r, \quad \varrho'_1, \dots, \varrho'_r$ given

Single-shot: $\exists\, \Phi$ CPTP: $\quad \Phi(\varrho_i) = \varrho'_i,\ i \in [r]\ \ ?$

Multi-copy: $\exists\, n \in \mathbb{N}$, Φ CPTP: $\quad \Phi(\varrho_i^{\otimes n}) = \varrho_i'^{\otimes n},\ i \in [r]\ \ ?$

Catalytic: $\exists\, \gamma_i \in \mathcal{S}(\mathcal{H})_{++}$, Φ CPTP: $\quad \Phi(\varrho_i \otimes \gamma_i) = \varrho'_i \otimes \gamma_i, \quad i \in [r]\,?$

Necessary: $\quad D_P^q$ r-variable monotone, additive quantum Rényi divergence

$$D_P^q((\varrho_i)_{i\in[r]}) \geq D_P^q((\varrho'_i)_{i\in[r]})$$

State conversion

$\varrho_1, \dots, \varrho_r, \quad \varrho'_1, \dots, \varrho'_r$ given

Single-shot: $\exists\ \Phi$ CPTP: $\quad \Phi(\varrho_i) = \varrho'_i,\ i \in [r]\ \ ?$

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Catalytic: $\exists\, \gamma_i \in \mathcal{S}(\mathcal{H})_{++}$, Φ CPTP: $\quad \Phi(\varrho_i \otimes \gamma_i) = \varrho'_i \otimes \gamma_i, \quad i \in [r]\,?$

Approximate catalytic:

$\forall \varepsilon > 0\, \exists\, \gamma_{i,\varepsilon}$, Φ CPTP: $\quad \Phi(\varrho_i \otimes \gamma_{i,\varepsilon}) = \varrho'_{i,\varepsilon} \otimes \gamma_{i,\varepsilon},\ \ \varrho'_{i,\varepsilon} \approx_\varepsilon \varrho'_i, \quad i \in [r]\,?$

Necessary: $\quad D_P^q$ r-variable monotone, additive, l.s.c. quantum Rényi divergence

$$D_P^q((\varrho_i)_{i\in[r]}) \geq D_P^q((\varrho'_i)_{i\in[r]})$$

State conversion

$\varrho_1, \dots, \varrho_r, \quad \varrho'_1, \dots, \varrho'_r$ given

Single-shot: $\exists\, \Phi$ CPTP: $\quad \Phi(\varrho_i) = \varrho'_i,\ i \in [r]$?

Multi-copy: $\exists\, n \in \mathbb{N}$, Φ CPTP: $\quad \Phi(\varrho_i^{\otimes n}) = \varrho_i'^{\otimes n},\ i \in [r]$?

Catalytic: $\exists\, \gamma_i \in \mathcal{S}(\mathcal{H})_{++}$, Φ CPTP: $\quad \Phi(\varrho_i \otimes \gamma_i) = \varrho'_i \otimes \gamma_i,\quad i \in [r]$?

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Necessary: $\quad D_P^q$ r-variable monotone, additive, l.s.c. quantum Rényi divergence

$$D_P^q((\varrho_i)_{i\in[r]}) \geq D_P^q((\varrho'_i)_{i\in[r]})$$

Classical case: (strict) inequality for all monotone multi-variate Rényi divergence is also sufficient.

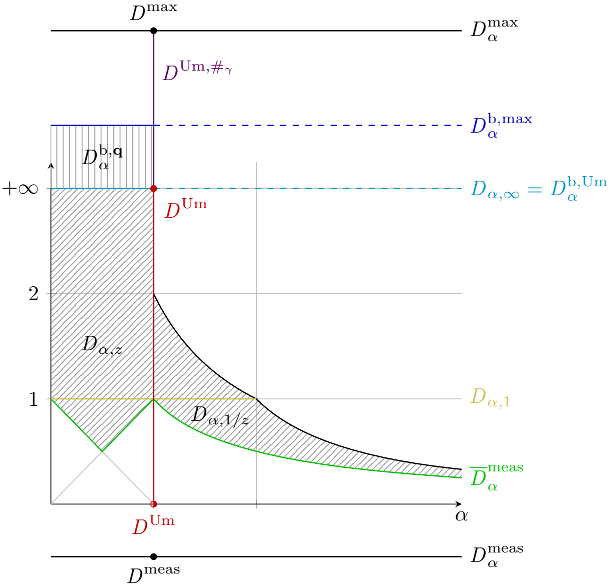
(Farooq, Fritz, Haapasalo, Tomamichel 2023)

Goal

1. Find quantum Rényi divergences (binary as well as multi-variate) with good mathematical properties.

2. Connect them to trade-off relations/state convertibility problems.

Quantum Rényi divergences



$$D^{\max}(\varrho\|\sigma)$$
$$= \mathrm{Tr}\,\varrho \log(\varrho^{\frac{1}{2}}\sigma^{-1}\varrho^{\frac{1}{2}})$$

Belavkin-Staszewski 1982

increasing (Araki-Lieb-Thirring inequality)

$$\lim_{\alpha\to 1} D_{\alpha,z} = D^{\mathrm{Um}}$$

$$D^{\mathrm{Um}}(\varrho\|\sigma) = \mathrm{Tr}\,\varrho(\log\varrho - \log\sigma)$$

Umegaki relative entropy

Geometric relative entropies

D^q quantum relative entropy

γ-weighted Kubo-Ando geometric mean $\gamma \in (0,1)$

$$\sigma \#_\gamma \varrho := \sigma^{1/2}(\sigma^{-1/2}\varrho\sigma^{-1/2})^\gamma \sigma^{1/2}$$

Geometric relative entropy:

$$D^{q,\#_\gamma}(\varrho\|\sigma) := \frac{1}{1-\gamma} D^q\left(\varrho\|\sigma\#_\gamma\varrho\right)$$

Geometric relative entropies

D^q quantum relative entropy

γ-weighted Kubo-Ando geometric mean $\qquad \gamma \in (0,1)$

$$\sigma \#_\gamma \varrho := \sigma^{1/2}(\sigma^{-1/2}\varrho\sigma^{-1/2})^\gamma \sigma^{1/2}$$

Geometric relative entropy:

$$D^{q,\#_\gamma}(\varrho \| \sigma) := \frac{1}{1-\gamma} D^q\left(\varrho \| \sigma \#_\gamma \varrho\right)$$

D^q additive $\Longrightarrow$ so is $D^{q,\#_\gamma}$

D^q monotone $\Longrightarrow$ so is $D^{q,\#_\gamma}$ $\qquad (\omega_1 \leq \omega_2 \quad \Longrightarrow \quad D^q(\tau\|\omega_1) \geq D^q(\tau\|\omega_2))$

Geometric relative entropies

Monotonicity:

$$D^q(\Phi(\varrho)\|\Phi(\sigma)) \leq D^q(\varrho\|\sigma), \qquad \omega_1 \leq \omega_2 \implies D^q(\tau\|\omega_1) \geq D^q(\tau\|\omega_2)$$

$$\begin{aligned} D^{q,\#_\gamma}(\Phi(\varrho)\|\Phi(\sigma)) &= \frac{1}{1-\gamma} D^q(\Phi(\varrho)\| \underbrace{\Phi(\sigma)\#_\gamma\Phi(\varrho)}_{\geq \Phi(\sigma\#_\gamma\varrho)}) \\ &\leq \frac{1}{1-\gamma} D^q(\Phi(\varrho)\|\Phi(\sigma\#_\gamma\varrho)) \\ &\leq \frac{1}{1-\gamma} D^q(\varrho\|\sigma\#_\gamma\varrho), \end{aligned}$$

Monotonicity:

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Cor.: $D^{\mathrm{Um}} \le D^{q,\#_\gamma} \le D^{\max}$

Geometric relative entropies

Monotonicity:

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Cor.: $D^{\mathrm{Um}} \le D^{\mathrm{Um},\#_\gamma} \le D^{q,\#_\gamma} \le D^{\max,\#_\gamma} \le D^{\max}$

Monotonicity:

$$D^q(\Phi(\varrho)\|\Phi(\sigma)) \le D^q(\varrho\|\sigma), \qquad \omega_1 \le \omega_2 \implies D^q(\tau\|\omega_1) \ge D^q(\tau\|\omega_2)$$

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Cor.: $D^{\mathrm{Um}} = \overline{D}^{\mathrm{meas}} \le D^{q,\#_\gamma} \le D^{\max}$

Cor.: $D^{\mathrm{Um}} \le D^{\mathrm{Um},\#_\gamma} \le D^{q,\#_\gamma} \le D^{\max,\#_\gamma} = D^{\max}$

Geometric relative entropies

Monotonicity:

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Cor.: $\gamma \mapsto D^{\mathrm{Um},\#_\gamma}(\varrho\|\sigma)$ increasing

Monotonicity:

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Cor.: $\gamma \mapsto D^{\mathrm{Um},\#_\gamma}(\varrho\|\sigma)$ increasing

Thm.: $$\lim_{\gamma\searrow 0} D^{\mathrm{Um},\#_\gamma}(\varrho\|\sigma) = D^{\mathrm{Um}}(\varrho\|\sigma)$$
$$\lim_{\gamma\nearrow 1} D^{\mathrm{Um},\#_\gamma}(\varrho\|\sigma) = D^{\max}(\varrho\|\sigma), \qquad \varrho,\sigma > 0$$

Monotonicity:

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Cor.: $D^{\mathrm{Um}} = \overline{D}^{\mathrm{meas}} \le D^{q,\#_\gamma} \le D^{\max}$

Cor.: $D^{\mathrm{Um}} \le D^{\mathrm{Um},\#_\gamma} \le D^{q,\#_\gamma} \le D^{\max,\#_\gamma} = D^{\max}$

Cor.: $\gamma \mapsto D^{\mathrm{Um},\#_\gamma}(\varrho\|\sigma)$ increasing

Thm.:

$$\lim_{\gamma\searrow 0} D^{q,\#_\gamma}(\varrho\|\sigma) = D^q(\varrho\|\sigma)$$

$$\lim_{\gamma\nearrow 1} D^{q,\#_\gamma}(\varrho\|\sigma) = D^{\max}(\varrho\|\sigma), \qquad \varrho,\sigma > 0$$

ψ unit vector, not an eigenvector of $\sigma > 0$

$$\begin{aligned} D^{\mathrm{Um}}(|\psi\rangle\langle\psi| \, \|\sigma) &= \big\langle \psi, (\log \sigma^{-1})\psi \big\rangle \\ &< \log \big\langle \psi, \sigma^{-1}\psi \big\rangle = D^{\mathrm{Um},\#_\gamma}(|\psi\rangle\langle\psi| \, \|\sigma) = D^{\max}(|\psi\rangle\langle\psi| \, \|\sigma) \end{aligned}$$

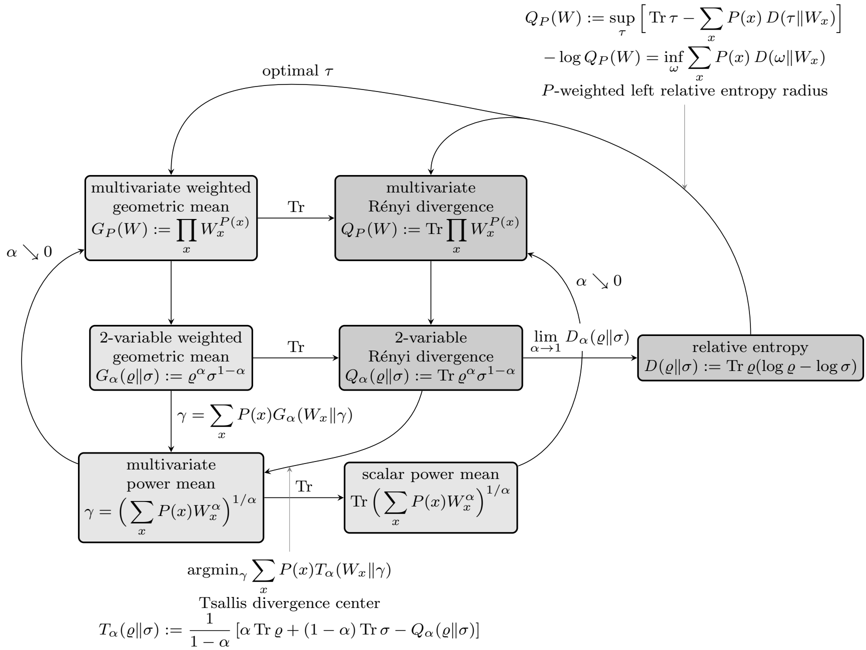
ψ unit vector, not an eigenvector of $\sigma > 0$

$$D^{\mathrm{Um}}(|\psi\rangle\langle\psi|\,\|\sigma) = \left\langle\psi, (\log\sigma^{-1})\psi\right\rangle$$
$$< \log\left\langle\psi, \sigma^{-1}\psi\right\rangle = D^{\mathrm{Um},\#_\gamma}(|\psi\rangle\langle\psi|\,\|\sigma) = D^{\max}(|\psi\rangle\langle\psi|\,\|\sigma)$$

Cor.: $\forall\, t, \gamma \in (0,1)\ \ \exists\, \varrho, \sigma :$

$$(1-t)D^{\mathrm{Um}}(\varrho\|\sigma) + tD^{\max}(\varrho\|\sigma) < D^{\mathrm{Um},\#_\gamma}(\varrho\|\sigma)$$

Multi-variate Rényi divergences and geometric means



Barycentric Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b},\mathbf{q}}(W) := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{q_x}(\tau\|W_x)\right\},$$

$$-\log Q_P^{\mathrm{b},\mathbf{q}}(W) = \inf_{\omega\in\mathcal{S}(\mathcal{H})}\sum_{x\in\mathcal{X}} P(x) D^{q_x}(\omega\|W_x) \qquad =: R_{D^{\mathbf{q}}}^{\mathrm{left}}(W,P)$$

Barycentric Rényi divergences

Barycentric Rényi divergence:

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- $Q_P^{\mathrm{b},\mathbf{q}}(W)$, $\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in P

Barycentric Rényi divergences

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- $Q_P^{\mathrm{b},\mathbf{q}}(W)$, $\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in P
- $P \geq 0$, all D^{q_x} monotone under Φ $\Longrightarrow$ so is $-\log Q_P^{\mathrm{b},\mathbf{q}}$

Barycentric Rényi divergences

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$$-\log Q_P^{\mathrm{b},\mathbf{q}}(W) = \inf_{\omega\in\mathcal{S}(\mathcal{H})} \sum_{x\in\mathcal{X}} P(x) D^{q_x}(\omega\|W_x) \qquad =: R_{D^{\mathbf{q}}}^{\mathrm{left}}(W,P)$$

- $Q_P^{\mathrm{b},\mathbf{q}}(W)$, $\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in P
- $P \geq 0$, all D^{q_x} monotone under Φ $\Longrightarrow$ so is $-\log Q_P^{\mathrm{b},\mathbf{q}}$
- $P \geq 0$, all D^{q_x} jointly convex $\Longrightarrow$ $-\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in W

Barycentric Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b},\mathbf{q}}(W) := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{q_x}(\tau\|W_x)\right\},$$

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- $Q_P^{\mathrm{b},\mathbf{q}}(W)$, $\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in P
- $P \geq 0$, all D^{q_x} monotone under Φ $\implies$ so is $-\log Q_P^{\mathrm{b},\mathbf{q}}$
- $P \geq 0$, all D^{q_x} jointly convex $\implies$ $-\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in W
- $P \geq 0$, all D^{q_x} jointly l.s.c $\implies$ $W \mapsto -\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ l.s.c.

Barycentric Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b},\mathbf{q}}(W) := \sup_{\tau \in \mathcal{B}(\mathcal{H})_+} \left\{ \operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{q_x}(\tau \| W_x) \right\},$$

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- $Q_P^{\mathrm{b},\mathbf{q}}(W)$, $\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in P
- $P \geq 0$, all D^{q_x} monotone under Φ $\Longrightarrow$ so is $-\log Q_P^{\mathrm{b},\mathbf{q}}$
- $P \geq 0$, all D^{q_x} jointly convex $\Longrightarrow$ $-\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ convex in W
- $P \geq 0$, all D^{q_x} jointly l.s.c $\Longrightarrow$ $W \mapsto -\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ l.s.c.
- all D^{q_x} additive $\Longrightarrow$ $-\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ subadditive

$$R_{D^{\mathbf{q}}}^{\mathrm{left}}((W_x^{(1)}\otimes W_x^{(2)})_{x\in\mathcal{X}},P) \leq R_{D^{\mathbf{q}}}^{\mathrm{left}}((W_x^{(1)})_{x\in\mathcal{X}},P) + R_{D^{\mathbf{q}}}^{\mathrm{left}}((W_x^{(2)})_{x\in\mathcal{X}},P)$$

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b},\mathbf{q}}(W) := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+} \left\{ \operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{q_x}(\tau\|W_x) \right\},$$

$$-\log Q_P^{\mathrm{b},\mathbf{q}}(W) = \inf_{\omega\in\mathcal{S}(\mathcal{H})} \sum_{x\in\mathcal{X}} P(x) D^{q_x}(\omega\|W_x) \qquad =: R_{D^{\mathbf{q}}}^{\mathrm{left}}(W,P)$$

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- $P \geq 0$, all D^{q_x} jointly l.s.c $\implies$ $W \mapsto -\log Q_P^{\mathrm{b},\mathbf{q}}(W)$ l.s.c.
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Additivity?

Barycentric Umegaki Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b,Um}} := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\tau\|W_x)\right\},$$

$$-\log Q_P^{\mathrm{b,Um}} = \inf_{\omega\in\mathcal{S}(\mathcal{H})}\sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\omega\|W_x) =: R_{D^{\mathrm{Um}}}^{\mathrm{left}}(W,P)$$

Barycentric Umegaki Rényi divergences

Barycentric Rényi divergence:

$$
\begin{aligned}
Q_P^{\mathrm{b,Um}} &:= \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau-\sum_{x\in\mathcal{X}}P(x)D^{\mathrm{Um}}(\tau\|W_x)\right\},\\
-\log Q_P^{\mathrm{b,Um}} &= \inf_{\omega\in\mathcal{S}(\mathcal{H})}\sum_{x\in\mathcal{X}}P(x)D^{\mathrm{Um}}(\omega\|W_x) \qquad =: R^{\mathrm{left}}_{D^{\mathrm{Um}}}(W,P)\\
&= -\log\operatorname{Tr}\underbrace{\exp\left(\sum\nolimits_x P(x)\log W_x\right)}_{=\tau_{opt}},
\end{aligned}
$$

Barycentric Umegaki Rényi divergences

Barycentric Rényi divergence:

$$
\begin{aligned}
Q_P^{\mathrm{b,Um}} &:= \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\tau\|W_x)\right\}, \\
-\log Q_P^{\mathrm{b,Um}} &= \inf_{\omega\in\mathcal{S}(\mathcal{H})} \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\omega\|W_x) \qquad =: R^{\mathrm{left}}_{D^{\mathrm{Um}}}(W,P) \\
&= -\log \operatorname{Tr} \underbrace{\exp\left(\sum\nolimits_x P(x)\log W_x\right)}_{=\tau_{opt}},
\end{aligned}
$$

Additivity:

$$
R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(1)}\otimes W_x^{(2)})_{x\in\mathcal{X}},P) = R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(1)})_{x\in\mathcal{X}},P) + R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(2)})_{x\in\mathcal{X}},P)
$$

Barycentric Umegaki Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b,Um}} := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\tau\|W_x)\right\},$$

$$-\log Q_P^{\mathrm{b,Um}} = \inf_{\omega\in\mathcal{S}(\mathcal{H})}\sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\omega\|W_x) =: R^{\mathrm{left}}_{D^{\mathrm{Um}}}(W,P)$$

$$= -\log \operatorname{Tr}\underbrace{\exp\left(\sum\nolimits_x P(x)\log W_x\right)}_{=\tau_{opt}},$$

Additivity:

$$R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(1)}\otimes W_x^{(2)})_{x\in\mathcal{X}},P) = R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(1)})_{x\in\mathcal{X}},P) + R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(2)})_{x\in\mathcal{X}},P)$$

Holevo quantity:

$$R^{\mathrm{right}}_{D^{\mathrm{Um}}}(W,P) := \inf_{\omega\in\mathcal{S}(\mathcal{H})}\sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(W_x\|\omega)$$

Barycentric Umegaki Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b,Um}} := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+}\left\{\operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\tau\|W_x)\right\},$$

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$$= -\log \operatorname{Tr} \underbrace{\exp\left(\sum_x P(x)\log W_x\right)}_{=\tau_{opt}},$$

Additivity:

$$R_{D^{\mathrm{Um}}}^{\mathrm{left}}((W_x^{(1)}\otimes W_x^{(2)})_{x\in\mathcal{X}},P) = R_{D^{\mathrm{Um}}}^{\mathrm{left}}((W_x^{(1)})_{x\in\mathcal{X}},P) + R_{D^{\mathrm{Um}}}^{\mathrm{left}}((W_x^{(2)})_{x\in\mathcal{X}},P)$$

Holevo quantity:

$$R_{D^{\mathrm{Um}}}^{\mathrm{right}}(W,P) := \inf_{\omega\in\mathcal{S}(\mathcal{H})}\sum_{x\in\mathcal{X}} P(x)D^{\mathrm{Um}}(W_x\|\omega) = \sum_{x\in\mathcal{X}} P(x)D^{\mathrm{Um}}\left(W_x\Big\|\sum_x P(x)W_x\right)$$

Barycentric Umegaki Rényi divergences

Barycentric Rényi divergence:

$$Q_P^{\mathrm{b,Um}} := \sup_{\tau\in\mathcal{B}(\mathcal{H})_+} \left\{ \operatorname{Tr}\tau - \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\tau\|W_x) \right\},$$

$$-\log Q_P^{\mathrm{b,Um}} = \inf_{\omega\in\mathcal{S}(\mathcal{H})} \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(\omega\|W_x) =: R^{\mathrm{left}}_{D^{\mathrm{Um}}}(W,P)$$

$$= -\log \operatorname{Tr} \underbrace{\exp\left(\sum\nolimits_x P(x)\log W_x\right)}_{=\tau_{opt}},$$

Additivity:

$$R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(1)}\otimes W_x^{(2)})_{x\in\mathcal{X}},P) = R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(1)})_{x\in\mathcal{X}},P) + R^{\mathrm{left}}_{D^{\mathrm{Um}}}((W_x^{(2)})_{x\in\mathcal{X}},P)$$

Holevo quantity:

$$R^{\mathrm{right}}_{D^{\mathrm{Um}}}(W,P) := \inf_{\omega\in\mathcal{S}(\mathcal{H})} \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}(W_x\|\omega) = \sum_{x\in\mathcal{X}} P(x) D^{\mathrm{Um}}\left(W_x\middle\| \sum\nolimits_x P(x)W_x\right)$$

$$R^{\mathrm{right}}_{D^{\mathrm{Um}}}((W_{x_1}^{(1)}\otimes W_{x_2}^{(2)})_{x_1,x_2},P_1\otimes P_2) = R^{\mathrm{right}}_{D^{\mathrm{Um}}}(W^{(1)},P_1) + R^{\mathrm{right}}_{D^{\mathrm{Um}}}(W^{(2)},P_2)$$

2-variable barycentric Rényi divergence

$$Q_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \sup_{\tau\in\mathcal{B}(\varrho^0\mathcal{H})_{\geq 0}} \{\operatorname{Tr}\tau - \alpha D^{q_0}(\tau\|\varrho) - (1-\alpha)D^{q_1}(\tau\|\sigma)\}$$

$$D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \frac{\psi_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) - \psi_1^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)}{\alpha-1}$$

$$= \frac{1}{1-\alpha}\inf_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})}\{\alpha D^{q_0}(\omega\|\varrho) + (1-\alpha)D^{q_1}(\omega\|\sigma)\} - \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho$$

2-variable barycentric Rényi divergence

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$$D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \frac{\psi_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) - \psi_1^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)}{\alpha-1}$$

$$= \frac{1}{1-\alpha}\inf_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})}\{\alpha D^{q_0}(\omega\|\varrho) + (1-\alpha)D^{q_1}(\omega\|\sigma)\} - \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho$$

- $\log Q_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)$ convex in α, $D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)$ monotone in α

$$\frac{1}{\operatorname{Tr}\varrho}D^{q_1}(\varrho\|\sigma) = \lim_{\alpha\nearrow 1} D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)$$

$$\frac{1}{\operatorname{Tr}\varrho}D^{q_0}(\varrho\|\sigma) = \lim_{\alpha\searrow 0}\frac{1}{\alpha}\left[(1-\alpha)D_\alpha^{\mathrm{b},\mathbf{q}}(\sigma\|\varrho) + \log\operatorname{Tr}\varrho - \log\operatorname{Tr}\sigma\right]$$

$$\lim_{\alpha\to+\infty} D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) = \sup_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})}\{D^{q_1}(\omega\|\sigma) - D^{q_0}(\omega\|\varrho)\}.$$

- Different (D^{q_0}, D^{q_1}) pairs generate different barycentric families.

Finiteness and non-barycentric Rényi divergences

$$Q_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \sup_{\tau\in\mathcal{B}(\varrho^0\mathcal{H})_{\geq 0}} \{\operatorname{Tr}\tau - \alpha D^{q_0}(\tau\|\varrho) - (1-\alpha)D^{q_1}(\tau\|\sigma)\}$$

$$D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \frac{\psi_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) - \psi_1^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)}{\alpha-1}$$
$$= \frac{1}{1-\alpha}\inf_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})}\{\alpha D^{q_0}(\omega\|\varrho) + (1-\alpha)D^{q_1}(\omega\|\sigma)\} - \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho$$

- Finiteness:

$$D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) = +\infty \begin{cases} \iff \varrho^0\wedge\sigma^0 = 0, & \text{when } \alpha\in[0,1), \\ \iff \varrho^0 \not\leq \sigma^0, & \text{when } \alpha > 1. \end{cases}$$

$$Q_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \sup_{\tau\in\mathcal{B}(\varrho^0\mathcal{H})_{\geq 0}} \{\operatorname{Tr}\tau - \alpha D^{q_0}(\tau\|\varrho) - (1-\alpha)D^{q_1}(\tau\|\sigma)\}$$

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- Finiteness:

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Cor.: $D_\alpha^q(\varrho\|\sigma) = +\infty \iff \varrho\perp\sigma$

$\Longrightarrow$ D^q is not a barycentric Rényi divergence

Finiteness and non-barycentric Rényi divergences

$$Q_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \sup_{\tau\in\mathcal{B}(\varrho^0\mathcal{H})_{\geq 0}} \{\operatorname{Tr}\tau - \alpha D^{q_0}(\tau\|\varrho) - (1-\alpha)D^{q_1}(\tau\|\sigma)\}$$

$$D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) := \frac{\psi_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) - \psi_1^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma)}{\alpha-1}$$

$$= \frac{1}{1-\alpha}\inf_{\omega\in\mathcal{S}(\varrho^0\mathcal{H})}\{\alpha D^{q_0}(\omega\|\varrho) + (1-\alpha)D^{q_1}(\omega\|\sigma)\} - \frac{1}{\alpha-1}\log\operatorname{Tr}\varrho$$

- Finiteness:

$$D_\alpha^{\mathrm{b},\mathbf{q}}(\varrho\|\sigma) = +\infty \begin{cases} \iff \varrho^0\wedge\sigma^0 = 0, & \text{when } \alpha\in[0,1), \\ \iff \varrho^0 \nleq \sigma^0, & \text{when } \alpha > 1. \end{cases}$$

Cor.: $D_\alpha^q(\varrho\|\sigma) = +\infty \iff \varrho\perp\sigma$

$\implies D^q$ is not a barycentric Rényi divergence

Cor.: D_α^{meas}, $D_{\alpha,z}$, $\alpha\in(0,1)$, $z\in(0,+\infty)$

are not barycentric Rényi divergences

Ordering and non-barycentric Rényi divergence

D^{q_0} and D^{q_1} CPTP-monotone

$$D_\alpha^{\mathrm{meas}} \lneq D_\alpha^{\mathrm{b,meas}} \leq D_\alpha^{\mathrm{b},\mathbf{q}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\mathrm{max}}, \qquad \alpha \in (0,1)$$

Ordering and non-barycentric Rényi divergence

D^{q_0} and D^{q_1} CPTP-monotone

$$D_\alpha^{\mathrm{meas}} \lneq D_\alpha^{\mathrm{b,meas}} \leq D_\alpha^{\mathrm{b},\mathbf{q}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\mathrm{max}}, \qquad \alpha \in (0,1)$$

D^{q_0} and D^{q_1} also additive

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \leq D_\alpha^{\mathrm{b},\mathbf{q}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\mathrm{max}}, \qquad \alpha \in (0,1)$$

Ordering and non-barycentric Rényi divergence

D^{q_0} and D^{q_1} CPTP-monotone

$$D_\alpha^{\mathrm{meas}} \lneq D_\alpha^{\mathrm{b,meas}} \leq D_\alpha^{\mathrm{b,\mathbf{q}}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

D^{q_0} and D^{q_1} also additive

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \leq D_\alpha^{\mathrm{b,\mathbf{q}}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

$D^{\mathrm{Um}} < D^{q_0}$ or $D^{\mathrm{Um}} < D^{q_1}$

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \lneq D_\alpha^{\mathrm{b,\mathbf{q}}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

Ordering and non-barycentric Rényi divergence

D^{q_0} and D^{q_1} CPTP-monotone

$$D_\alpha^{\mathrm{meas}} \lneq D_\alpha^{\mathrm{b,meas}} \le D_\alpha^{\mathrm{b},\mathbf{q}} \le D_\alpha^{\mathrm{b,max}} \le D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

D^{q_0} and D^{q_1} also additive

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \le D_\alpha^{\mathrm{b},\mathbf{q}} \le D_\alpha^{\mathrm{b,max}} \le D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

$D^{\mathrm{Um}} < D^{q_0}$ or $D^{\mathrm{Um}} < D^{q_1}$

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \lneq D_\alpha^{\mathrm{b},\mathbf{q}} \le D_\alpha^{\mathrm{b,max}} \le D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

$D^{q_0} < D^{\max}$ or $D^{q_1} < D^{\max}$

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \le D_\alpha^{\mathrm{b},\mathbf{q}} \lneq D_\alpha^{\mathrm{b,max}} \le D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

Ordering and non-barycentric Rényi divergence

D^{q_0} and D^{q_1} CPTP-monotone

$$D_\alpha^{\mathrm{meas}} \lneq D_\alpha^{\mathrm{b,meas}} \leq D_\alpha^{\mathrm{b},\mathbf{q}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

D^{q_0} and D^{q_1} also additive

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \leq D_\alpha^{\mathrm{b},\mathbf{q}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

$D^{\mathrm{Um}} < D^{q_0}$ or $D^{\mathrm{Um}} < D^{q_1}$

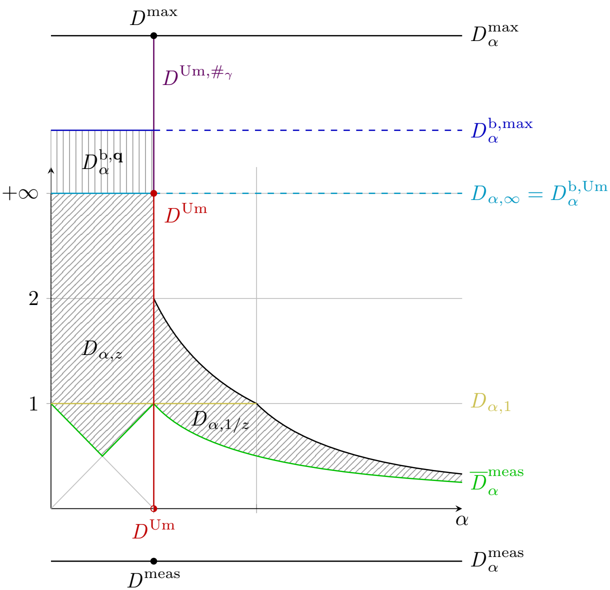
$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \lneq D_\alpha^{\mathrm{b},\mathbf{q}} \leq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

$D^{q_0} < D^{\max}$ or $D^{q_1} < D^{\max}$

$$D_{\alpha,+\infty} = D_\alpha^{\mathrm{b,Um}} \leq D_\alpha^{\mathrm{b},\mathbf{q}} \lneq D_\alpha^{\mathrm{b,max}} \leq D_\alpha^{\max}, \qquad \alpha \in (0,1)$$

Thm.: $D_\alpha^{\mathrm{b,max}} < D_\alpha^{\max}$, $\alpha \in (0,1)$.

Quantum Rényi divergences



$$D^{\max}(\varrho\|\sigma) = \operatorname{Tr} \varrho \log(\varrho^{-\frac{1}{2}} \sigma \varrho^{-\frac{1}{2}})$$

↑ increasing (Araki-Lieb-Thirring inequality)

$$\lim_{\alpha \to 1} D_{\alpha,z} = D^{\mathrm{Um}}$$

$$D^{\mathrm{Um}}(\varrho\|\sigma) = \operatorname{Tr} \varrho(\log \varrho - \log \sigma)$$

Umegaki relative entropy

$D_\alpha^{\mathrm{b,max}} < D_\alpha^{\max}$

Lemma:

$$
\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma\#_\alpha\varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\Big\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\Big\|\sigma\right)
\end{aligned}
$$

$D_\alpha^{\mathrm{b,max}} < D_\alpha^{\max}$

Lemma:

$$-\log Q_\alpha^{\max}(\varrho\|\sigma) = -\log \operatorname{Tr} \sigma\#_\alpha\varrho$$
$$= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\;\ldots\;)}\Big\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\;\ldots\;)}\Big\|\sigma\right)$$

Possible multi-variate generalization:

$$-\log \widetilde{Q}_P^{\max}(W) := \sum_x P(x) D^{\max}\left(\frac{G_P^{\max}(W)}{\operatorname{Tr} G_P^{\max}(W)}\Big\|W_x\right)$$

$D_\alpha^{\mathrm{b,max}} < D_\alpha^{\max}$

Lemma:

$$
\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma\#_\alpha\varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\Big\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\Big\|\sigma\right)
\end{aligned}
$$

Possible multi-variate generalization:

$$
\begin{aligned}
-\log \widetilde{Q}_P^{\max}(W) &:= \sum_x P(x) D^{\max}\left(\frac{G_P^{\max}(W)}{\operatorname{Tr} G_P^{\max}(W)}\Big\|W_x\right) \\
&= -\log Q_P^{\max}(W)\ ?
\end{aligned}
$$

$D_\alpha^{\mathrm{b,max}} < D_\alpha^{\max}$

Lemma:

$$
\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma\#_\alpha\varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\sigma\right)
\end{aligned}
$$

Lemma:

$$\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma\#_\alpha\varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\sigma\right) \\
&\geq \inf_{\omega\in\mathcal{S}(\mathcal{H})}\left\{\alpha D^{\max}(\omega\|\varrho) + (1-\alpha)D^{\max}(\omega\|\sigma)\right\}
\end{aligned}$$

$D_\alpha^{\mathrm{b,max}} < D_\alpha^{\max}$

Lemma:

$$
\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma\#_\alpha\varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\sigma\right) \\
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\end{aligned}
$$

Lemma: $\sigma^{-1/2}\varrho\sigma^{-1/2} = \sum_{i=1}^{r}\lambda_i P_i.$ $\quad \pi_{\mathcal{H}} := I/\dim\mathcal{H}$

$$
\partial_{\pi_{\mathcal{H}}} := \frac{d}{dt}\Big|_{t=0}\left[\alpha D^{\max}\left((1-t)\widehat{\sigma\#_\alpha\varrho} + t\pi_{\mathcal{H}}\middle\|\varrho\right) + (1-\alpha)D^{\max}\left((1-t)\widehat{\sigma\#_\alpha\varrho} + t\pi_{\mathcal{H}}\middle\|\sigma\right)\right]
$$

Lemma:

$$
\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma\#_\alpha\varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\varrho\right) + (1-\alpha)D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \ldots\ )}\middle\|\sigma\right) \\
&\geq \inf_{\omega\in\mathcal{S}(\mathcal{H})}\{\alpha D^{\max}(\omega\|\varrho) + (1-\alpha)D^{\max}(\omega\|\sigma)\}
\end{aligned}
$$

Lemma: $\sigma^{-1/2}\varrho\sigma^{-1/2} = \sum_{i=1}^{r}\lambda_i P_i.$ $\quad \pi_{\mathcal{H}} := I/\dim\mathcal{H}$

$$
\begin{aligned}
\partial_{\pi_{\mathcal{H}}} &:= \frac{d}{dt}\Big|_{t=0}\left[\alpha D^{\max}\left((1-t)\widehat{\sigma\#_\alpha\varrho} + t\pi_{\mathcal{H}}\|\varrho\right) + (1-\alpha)D^{\max}\left((1-t)\widehat{\sigma\#_\alpha\varrho} + t\pi_{\mathcal{H}}\|\sigma\right)\right] \\
&= -1 + \frac{1}{d}\sum\nolimits_{i,j}\operatorname{Tr} P_i\sigma P_j\sigma^{-1}\underbrace{(\log\lambda_i - \log\lambda_j)\frac{(\lambda_i-\lambda_j)}{(\lambda_i^\alpha-\lambda_j^\alpha)(\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha})}}_{=:\Lambda_{\alpha,i,j}>1}
\end{aligned}
$$

Lemma:

$$
\begin{aligned}
-\log Q_\alpha^{\max}(\varrho\|\sigma) &= -\log \operatorname{Tr} \sigma \#_\alpha \varrho \\
&= \alpha D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \dots\ )}\middle\|\varrho\right) + (1-\alpha) D^{\max}\left(\frac{\sigma\#_\alpha\varrho}{\operatorname{Tr}(\ \dots\ )}\middle\|\sigma\right) \\
&\geq \inf_{\omega\in\mathcal{S}(\mathcal{H})} \left\{\alpha D^{\max}(\omega\|\varrho) + (1-\alpha) D^{\max}(\omega\|\sigma)\right\}
\end{aligned}
$$

Lemma: $\sigma^{-1/2}\varrho\sigma^{-1/2} = \sum_{i=1}^r \lambda_i P_i$. $\quad \pi_{\mathcal{H}} := I/\dim\mathcal{H}$

$$
\begin{aligned}
\partial_{\pi_\mathcal{H}} &:= \frac{d}{dt}\Big|_{t=0}\left[\alpha D^{\max}\left((1-t)\widehat{\sigma\#_\alpha\varrho} + t\pi_\mathcal{H}\middle\|\varrho\right) + (1-\alpha)D^{\max}\left((1-t)\widehat{\sigma\#_\alpha\varrho} + t\pi_\mathcal{H}\middle\|\sigma\right)\right] \\
&= -1 + \tfrac{1}{d}\textstyle\sum_{i,j} \operatorname{Tr} P_i\sigma P_j\sigma^{-1} \underbrace{(\log\lambda_i - \log\lambda_j)\frac{(\lambda_i-\lambda_j)}{(\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha})}}_{=:\Lambda_{\alpha,i,j}>1} \\
&= -1 + \tfrac{1}{d}\textstyle\sum_{i,j} \underbrace{\langle e_i, \sigma e_j\rangle}_{=:S_{i,j}} \underbrace{\langle e_j, \sigma^{-1} e_i\rangle}_{=(S^{-1})^{\mathrm{T}}_{i,j}} \cdot \Lambda_{\alpha,i,j} = -1 + \left\langle u, (S\star(S^{-1})^{\mathrm{T}}\star\Lambda_\alpha)u\right\rangle,
\end{aligned}
$$

Lemma: $\partial_{\pi_{\mathcal{H}}} = -1 + \left\langle u, (S \star (S^{-1})^{\mathrm{T}} \star \Lambda_\alpha) u \right\rangle$

$$S = \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}, \qquad (S^{-1})^{\mathrm{T}} = \frac{4}{1-\|r\|^2} \cdot \frac{1}{2} \begin{bmatrix} 1-z & -x+iy \\ -x-iy & 1+z \end{bmatrix},$$

$$S \star (S^{-1})^{\mathrm{T}} = \frac{1}{1-\|r\|^2} \begin{bmatrix} 1-z^2 & -(x^2+y^2) \\ -(x^2+y^2) & 1-z^2 \end{bmatrix}.$$

where $r := (x,y,z) \in \mathbb{R}^3$ s.t. $\|r\|^2 = x^2+y^2+z^2 < 1$

$$\partial_{\pi_{\mathcal{H}}} = -1 + \frac{1}{1-\|r\|^2} \left[1 - z^2 - (x^2+y^2)\Lambda_{\alpha,1,2} \right] < 0$$

Conclusion

1. Rényi divergences are ubiquitous in (quantum) information theory.

2. Still far from completely understood.

3. Need for good multi-variate quantum Rényi divergences.

   Multi-variate Petz-type and sandwiched?

   Applications in information theoretic problems?