Quantum measurements constrained by the third law of thermodynamics

M. Hamed Mohammady and Takayuki Miyadera

Quantum TUT Workshop
22nd Feb 2024
Toyohashi University of Technology

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 801505

Outline

Outline

- Operational formulation of the third law for channels

Outline

- Operational formulation of the third law for channels
- Quantum measurements

Outline

- Operational formulation of the third law for channels
- Quantum measurements
- Measurements constrained by the third law

Operational formulation of the third law for channels

Formulation of the third law for channels

Formulation of the third law for channels

- The third law states that systems can't be cooled to absolute zero temperature

Formulation of the third law for channels

- The third law states that systems can't be cooled to absolute zero temperature
- At thermal equilibrium with temperature T, a finite quantum system with Hamiltonian H is in a Gibbs state $\tau(H,T) := e^{-\frac{H}{T}}/\mathrm{tr}[e^{-\frac{H}{T}}]$

Formulation of the third law for channels

- The third law states that systems can't be cooled to absolute zero temperature
- At thermal equilibrium with temperature T, a finite quantum system with Hamiltonian H is in a Gibbs state $\tau(H,T) := e^{-\frac{H}{T}}/\mathrm{tr}[e^{-\frac{H}{T}}]$
- $\tau(H,T)$ has full rank when $T > 0$, and low rank when $T = 0$

Formulation of the third law for channels

- The third law states that systems can't be cooled to absolute zero temperature
- At thermal equilibrium with temperature T, a finite quantum system with Hamiltonian H is in a Gibbs state $\tau(H,T) := e^{-\frac{H}{T}}/\mathrm{tr}[e^{-\frac{H}{T}}]$
- $\tau(H,T)$ has full rank when $T > 0$, and low rank when $T = 0$
- A channel $\Phi : \tau(H,T) \mapsto \tau(H,0)$ violates the third law

Formulation of the third law for channels

- The third law states that systems can't be cooled to absolute zero temperature
- At thermal equilibrium with temperature T, a finite quantum system with Hamiltonian H is in a Gibbs state $\tau(H,T) := e^{-\frac{H}{T}}/\mathrm{tr}[e^{-\frac{H}{T}}]$
- $\tau(H,T)$ has full rank when $T > 0$, and low rank when $T = 0$
- A channel $\Phi : \tau(H,T) \mapsto \tau(H,0)$ violates the third law

Definition. *A channel $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is constrained by the third law if for all states ρ on $\mathcal{H}$,*

$$\frac{\mathrm{rank}(\Phi(\rho))}{\dim(\mathcal{K})} \geqslant \frac{\mathrm{rank}(\rho)}{\dim(\mathcal{H})}$$

The third law and state preparations

The third law and state preparations

- State preparations on $\mathcal{H}$ are represented by channels

$$\mathcal{P}(\mathcal{H}) := \{\Phi : \mathcal{L}(\mathbb{C}^1) \to \mathcal{L}(\mathcal{H})\}$$

G. Gour and M. M. Wilde, IEEE International Symposium on Information Theory (2020)

The third law and state preparations

- State preparations on $\mathcal{H}$ are represented by channels

$$\mathcal{P}(\mathcal{H}) := \{\Phi : \mathcal{L}(\mathbb{C}^1) \to \mathcal{L}(\mathcal{H})\}$$

G. Gour and M. M. Wilde, IEEE International Symposium on Information Theory (2020)

- The only state on $\mathbb{C}^1 \equiv \mathbb{C}|\Omega\rangle$ is $|\Omega\rangle\langle\Omega|$, which has full-rank in $\mathbb{C}^1$

The third law and state preparations

- State preparations on $\mathcal{H}$ are represented by channels

$$\mathcal{P}(\mathcal{H}) := \{\Phi : \mathcal{L}(\mathbb{C}^1) \to \mathcal{L}(\mathcal{H})\}$$

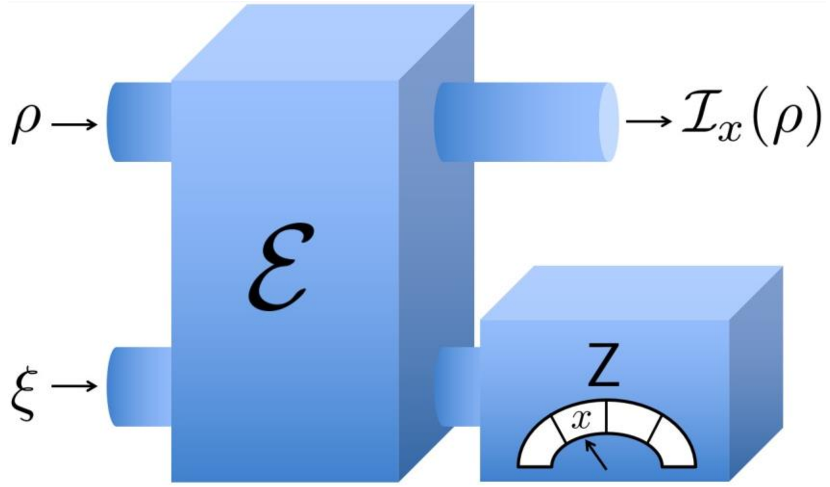
G. Gour and M. M. Wilde, IEEE International Symposium on Information Theory (2020)

- The only state on $\mathbb{C}^1 \equiv \mathbb{C}|\Omega\rangle$ is $|\Omega\rangle\langle\Omega|$, which has full-rank in $\mathbb{C}^1$

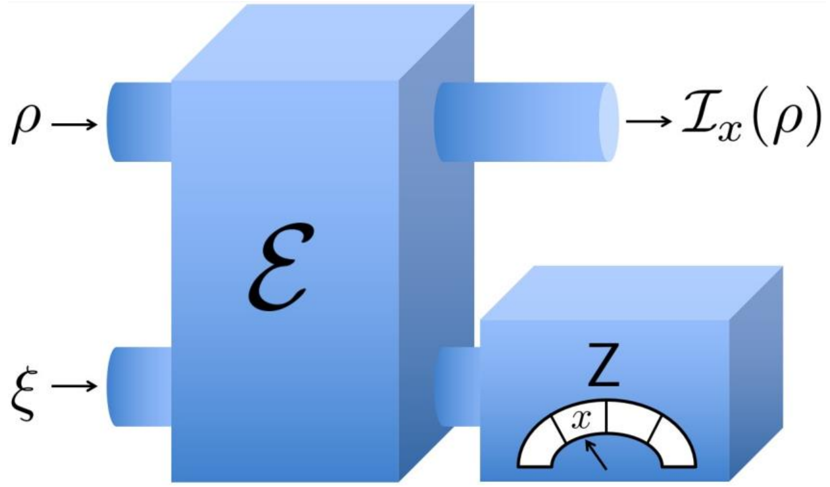
- Under the third law constraint, state preparations $\rho = \Phi(|\Omega\rangle\langle\Omega|)$ have full rank in $\mathcal{H}$

Quantum Measurements

Measurement schemes, instruments, and POVMs

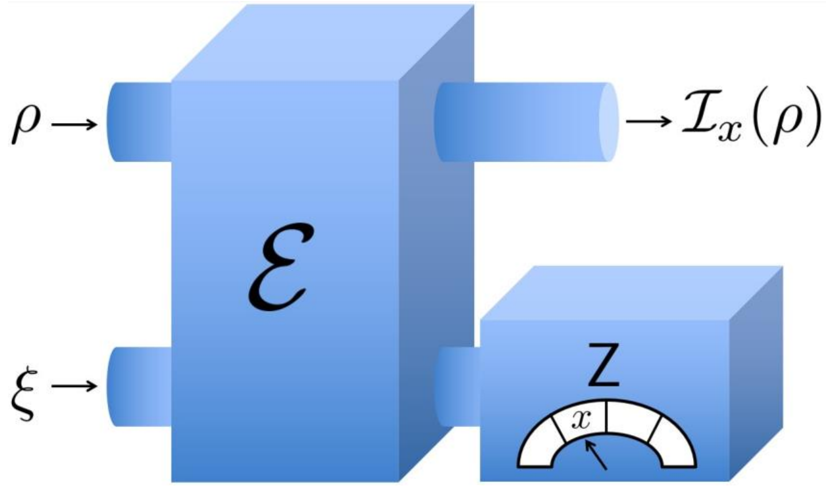


Measurement schemes, instruments, and POVMs



- $\mathcal{M} := (\mathcal{H}_\mathcal{A}, \xi, \mathcal{E}, \mathsf{Z})$ is a measurement scheme for an instrument $\mathcal{I} := \{\mathcal{I}_x : x \in \mathcal{X}\}$ compatible with an observable (POVM) $\mathsf{E} := \{\mathsf{E}_x : x \in \mathcal{X}\}$ acting in $\mathcal{H}_\mathcal{S}$

Measurement schemes, instruments, and POVMs



- $\mathcal{M} := (\mathcal{H}_\mathcal{A}, \xi, \mathcal{E}, \mathsf{Z})$ is a measurement scheme for an instrument $\mathcal{I} := \{\mathcal{I}_x : x \in \mathcal{X}\}$ compatible with an observable (POVM) $\mathsf{E} := \{\mathsf{E}_x : x \in \mathcal{X}\}$ acting in $\mathcal{H}_\mathcal{S}$

$$\mathcal{I}_x(\rho) = \mathrm{tr}_\mathcal{A}[\mathbb{1}_\mathcal{S} \otimes \mathsf{Z}_x \mathcal{E}(\rho \otimes \xi)] \, \forall x, \rho$$

M. Ozawa, J. Math. Phys. 25, 79 (1984)

$$\mathrm{tr}[\mathcal{I}_x(\rho)] = \mathrm{tr}[\mathsf{E}_x \rho] \, \forall x, \rho$$

Measurement schemes and the third law

Measurement schemes and the third law

Definition. *A measurement scheme* $\mathcal{M} := (\mathcal{H}_\mathcal{A}, \xi, \mathcal{E}, \mathsf{Z})$ *is constrained by the third law if the following hold:*

1. ξ *is a full-rank state on* $\mathcal{H}_\mathcal{A}$
2. $\mathcal{E}$ *is a rank non-decreasing channel acting in* $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A}$

Measurement schemes and the third law

Definition. *A measurement scheme* $\mathcal{M} := (\mathcal{H}_{\mathcal{A}}, \xi, \mathcal{E}, \mathsf{Z})$ *is constrained by the third law if the following hold:*

1. ξ *is a full-rank state on* $\mathcal{H}_{\mathcal{A}}$
2. $\mathcal{E}$ *is a rank non-decreasing channel acting in* $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}$

- The third law does not limit measurability of any observable; choosing $\mathcal{E}$ as a unitary swap channel, then $\mathsf{E} = \mathsf{Z}$

Measurement schemes and the third law

Definition. *A measurement scheme* $\mathcal{M} := (\mathcal{H}_\mathcal{A}, \xi, \mathcal{E}, \mathsf{Z})$ *is constrained by the third law if the following hold:*

1. ξ *is a full-rank state on* $\mathcal{H}_\mathcal{A}$
2. $\mathcal{E}$ *is a rank non-decreasing channel acting in* $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A}$

- The third law does not limit measurability of any observable; choosing $\mathcal{E}$ as a unitary swap channel, then $\mathsf{E} = \mathsf{Z}$
- But the third law limits the realisable instruments; as shown by Guryanova *et al.*, the standard von Neumann measurements $\rho \mapsto |\psi_x\rangle\langle\psi_x|\rho|\psi_x\rangle\langle\psi_x|$ are ruled out

Measurement schemes and the third law

Definition. *A measurement scheme* $\mathcal{M} := (\mathcal{H}_\mathcal{A}, \xi, \mathcal{E}, \mathsf{Z})$ *is constrained by the third law if the following hold:*

1. ξ *is a full-rank state on* $\mathcal{H}_\mathcal{A}$
2. $\mathcal{E}$ *is a rank non-decreasing channel acting in* $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A}$

- The third law does not limit measurability of any observable; choosing $\mathcal{E}$ as a unitary swap channel, then $\mathsf{E} = \mathsf{Z}$
- But the third law limits the realisable instruments; as shown by Guryanova *et al.*, the standard von Neumann measurements $\rho \mapsto |\psi_x\rangle\langle\psi_x|\rho|\psi_x\rangle\langle\psi_x|$ are ruled out

Y. Guryanova , N. Friis, M. Huber, Quantum 4, 222 (2019)

Measurements constrained by the third law

EPR criterion of physical reality

EPR criterion of physical reality

- The EPR criterion of physical reality reads:

EPR criterion of physical reality

- The EPR criterion of physical reality reads:

  "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"

A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935)

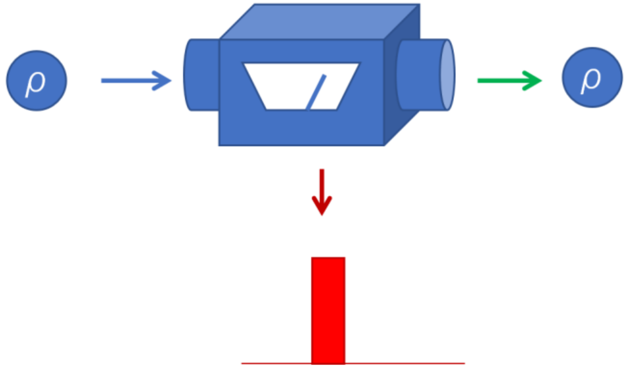
EPR criterion of physical reality

- The EPR criterion of physical reality reads:

"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"

A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935)

- The EPR criterion rests on the notion of "ideal" measurement



Ideal measurements

Ideal measurements

- $\mathcal{I}$ is an ideal measurement of E if for all x there exists ρ such that $\mathrm{tr}[\mathsf{E}_x\rho] = 1$, and if for every x and ρ the following implication holds:

$$\mathrm{tr}[\mathsf{E}_x\rho] = 1 \implies \mathcal{I}_x(\rho) = \rho$$

Ideal measurements

- $\mathcal{I}$ is an ideal measurement of E if for all x there exists ρ such that $\mathrm{tr}[\mathsf{E}_x\rho] = 1$, and if for every x and ρ the following implication holds:

$$\mathrm{tr}[\mathsf{E}_x\rho] = 1 \implies \mathcal{I}_x(\rho) = \rho$$

- Without any constraints, E admits an ideal measurement if and only if all of its effects have eigenvalue 1

Ideal measurements

- $\mathcal{I}$ is an ideal measurement of E if for all x there exists ρ such that $\mathrm{tr}[\mathsf{E}_x \rho] = 1$, and if for every x and ρ the following implication holds:

$$\mathrm{tr}[\mathsf{E}_x \rho] = 1 \implies \mathcal{I}_x(\rho) = \rho$$

- Without any constraints, E admits an ideal measurement if and only if all of its effects have eigenvalue 1

Theorem. *Under the third law constraint, no observable admits an ideal measurement*

Ideal measurements

- $\mathcal{I}$ is an ideal measurement of E if for all x there exists ρ such that $\mathrm{tr}[\mathsf{E}_x\rho] = 1$, and if for every x and ρ the following implication holds:

$$\mathrm{tr}[\mathsf{E}_x\rho] = 1 \implies \mathcal{I}_x(\rho) = \rho$$

- Without any constraints, E admits an ideal measurement if and only if all of its effects have eigenvalue 1

Theorem. *Under the third law constraint, no observable admits an ideal measurement*

$$\mathrm{tr}[\mathsf{E}_x\rho] = 1 \implies \mathrm{rank}(\mathcal{I}_x(\rho)) > \mathrm{rank}(\rho)$$

Approximately ideal measurements

Approximately ideal measurements

- The Lüders E-compatible instrument has the operations $\mathcal{I}_x^L(\cdot) := \sqrt{\mathsf{E}_x} \cdot \sqrt{\mathsf{E}_x}$

Approximately ideal measurements

- The Lüders E-compatible instrument has the operations $\mathcal{I}_x^L(\cdot) := \sqrt{\mathsf{E}_x} \cdot \sqrt{\mathsf{E}_x}$
- Lüders instruments are approximately ideal; for any $0 \leqslant \epsilon < 1$,

$$\mathrm{tr}[\mathsf{E}_x \rho] \geqslant 1 - \epsilon \implies \frac{1}{2} \| \rho - \mathcal{I}_x^L(\rho)/\mathrm{tr}[\mathsf{E}_x \rho] \|_1 \leqslant \sqrt{\epsilon}$$

P. Busch, Phys. Rev. D 33, 2253 (1986)
P. Busch and G. Jaeger, Found. Phys. 40, 1341 (2010)

Approximately ideal measurements

- The Lüders E-compatible instrument has the operations $\mathcal{I}_x^L(\cdot) := \sqrt{\mathsf{E}_x} \cdot \sqrt{\mathsf{E}_x}$
- Lüders instruments are approximately ideal; for any $0 \leqslant \epsilon < 1$,

$$\mathrm{tr}[\mathsf{E}_x \rho] \geqslant 1 - \epsilon \implies \frac{1}{2} \|\rho - \mathcal{I}_x^L(\rho)/\mathrm{tr}[\mathsf{E}_x \rho]\|_1 \leqslant \sqrt{\epsilon}$$

P. Busch, Phys. Rev. D 33, 2253 (1986)
P. Busch and G. Jaeger, Found. Phys. 40, 1341 (2010)

Theorem. *Under the third law constraint,* E *admits a Lüders instrument if and only if it none of the effects have eigenvalue 0 or 1*

Approximately ideal measurements

- The Lüders E-compatible instrument has the operations $\mathcal{I}_x^L(\cdot) := \sqrt{\mathsf{E}_x} \cdot \sqrt{\mathsf{E}_x}$
- Lüders instruments are approximately ideal; for any $0 \leqslant \epsilon < 1$,

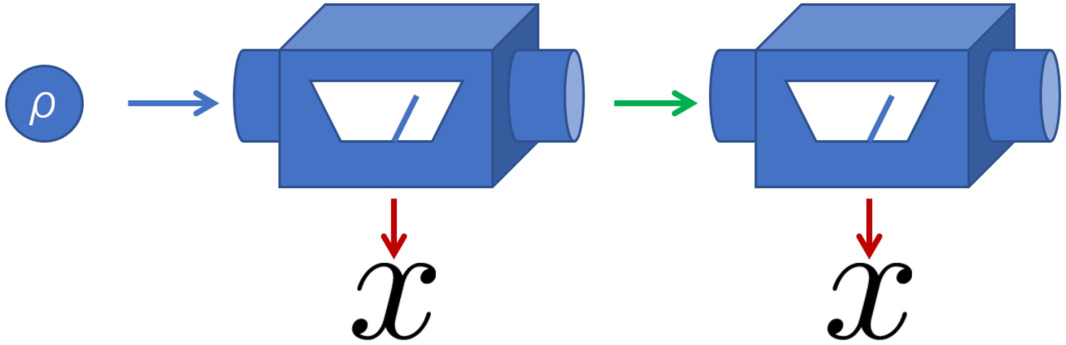
$$\mathrm{tr}[\mathsf{E}_x \rho] \geqslant 1 - \epsilon \implies \frac{1}{2} \| \rho - \mathcal{I}_x^L(\rho) / \mathrm{tr}[\mathsf{E}_x \rho] \|_1 \leqslant \sqrt{\epsilon}$$

P. Busch, Phys. Rev. D 33, 2253 (1986)
P. Busch and G. Jaeger, Found. Phys. 40, 1341 (2010)

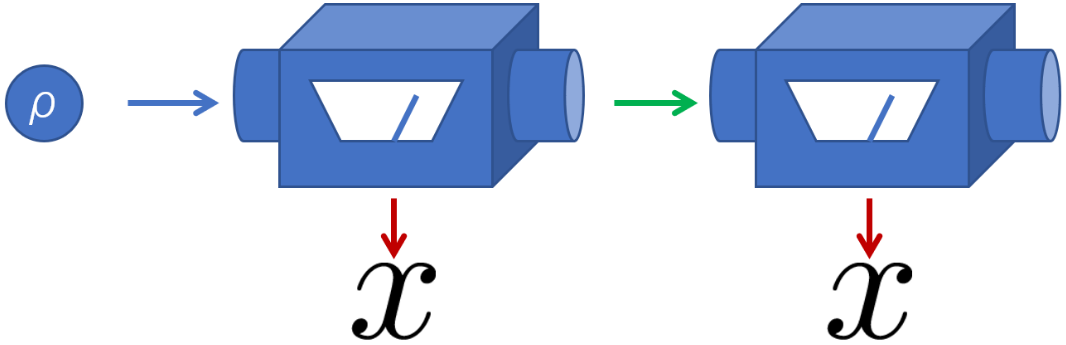
Theorem. *Under the third law constraint,* E *admits a Lüders instrument if and only if it none of the effects have eigenvalue 0 or 1*

- If events are indefinite in all states, then the third law is compatible with an approximate/unsharp variant of the EPR criteron

Repeatable measurements

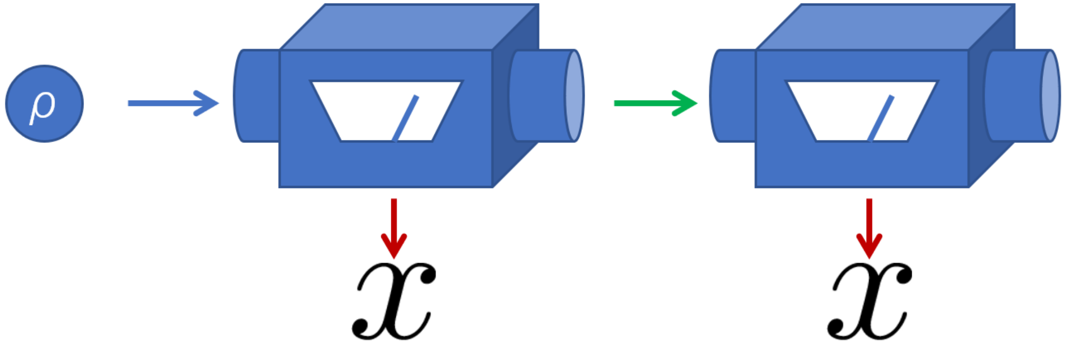


Repeatable measurements



- The repeatability principle of von Neumann reads:

Repeatable measurements

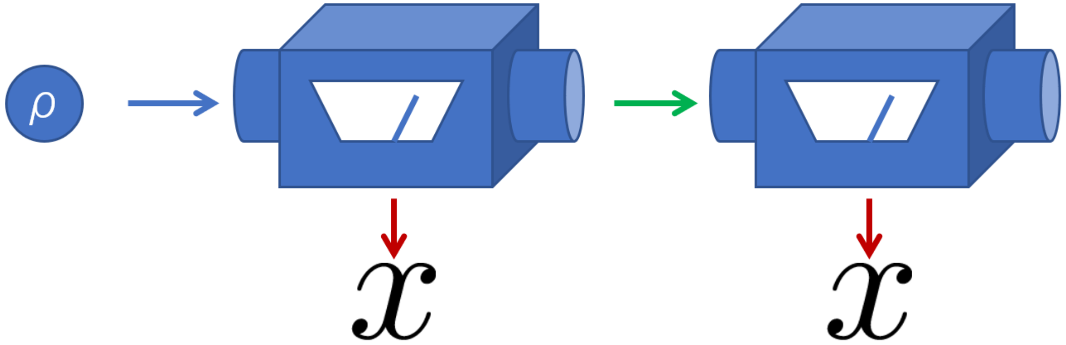


- The repeatability principle of von Neumann reads:

"If the physical quantity R is measured twice in prompt succession in a system S then we get the same value each time"

J. von Neumann, *Mathematical Foundations of Quantum Mechanics: New Edition* (2018), p. 218

Repeatable measurements



- The repeatability principle of von Neumann reads:

"If the physical quantity R is measured twice in prompt succession in a system S then we get the same value each time"

J. von Neumann, *Mathematical Foundations of Quantum Mechanics: New Edition* (2018), p. 218

- Repeatability and ideality coincide only for sharp rank-1 observables; in general a measurement may be repeatable but not ideal, or ideal but not repeatable

Repeatable measurements

Repeatable measurements

- $\mathcal{I}$ is a repeatable measurement of E if

$$\mathrm{tr}[\mathsf{E}_y \mathcal{I}_x(\rho)] = \delta_{x,y} \mathrm{tr}[\mathsf{E}_x \rho] \, \forall \rho, x, y$$

Repeatable measurements

- $\mathcal{I}$ is a repeatable measurement of E if

$$\mathrm{tr}[\mathsf{E}_y \mathcal{I}_x(\rho)] = \delta_{x,y} \mathrm{tr}[\mathsf{E}_x \rho] \, \forall \rho, x, y$$

- Without any constraints, E admits a repeatable measurement if and only if all of its effects have eigenvalue 1

Repeatable measurements

- $\mathcal{I}$ is a repeatable measurement of E if

$$\mathrm{tr}[\mathsf{E}_y \mathcal{I}_x(\rho)] = \delta_{x,y} \mathrm{tr}[\mathsf{E}_x \rho] \, \forall \rho, x, y$$

- Without any constraints, E admits a repeatable measurement if and only if all of its effects have eigenvalue 1

Theorem. *Under the third law constraint, no observable admits a repeatable measurement*

Repeatable measurements

- $\mathcal{I}$ is a repeatable measurement of E if

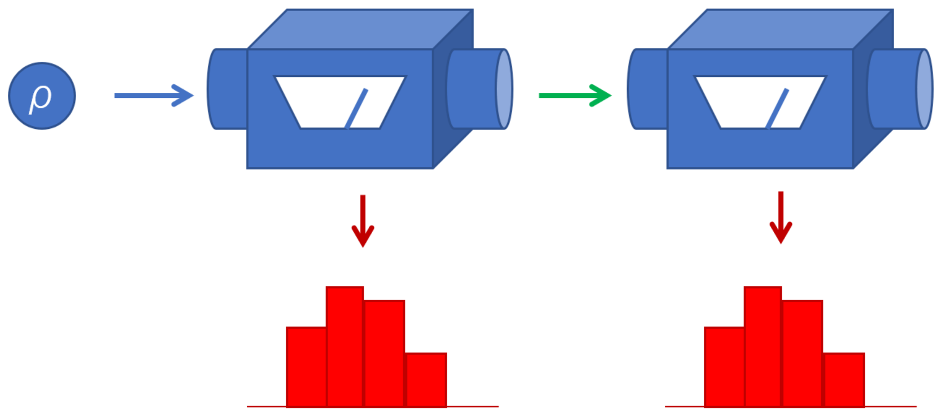
$$\mathrm{tr}[\mathsf{E}_y \mathcal{I}_x(\rho)] = \delta_{x,y} \mathrm{tr}[\mathsf{E}_x \rho] \, \forall \rho, x, y$$

- Without any constraints, E admits a repeatable measurement if and only if all of its effects have eigenvalue 1

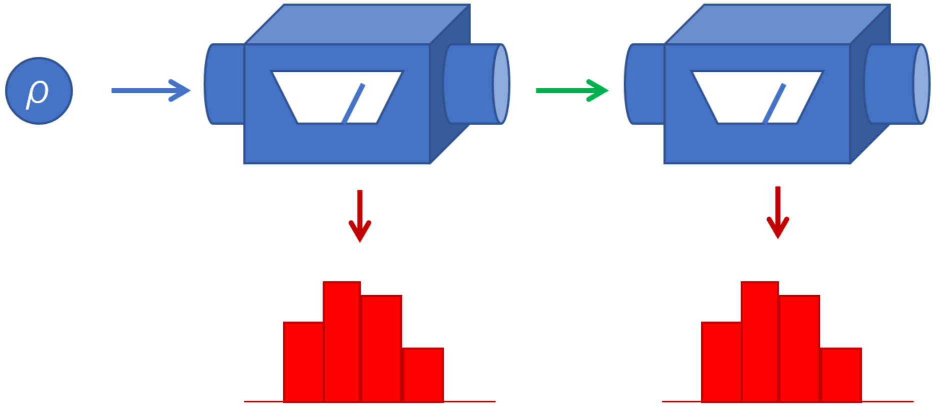
Theorem. *Under the third law constraint, no observable admits a repeatable measurement*

$$\text{For every full-rank state } \rho, \ \mathrm{tr}[\mathsf{E}_y \mathcal{I}_x(\rho)] > 0 \, \forall x, y$$

Measurements of the first kind



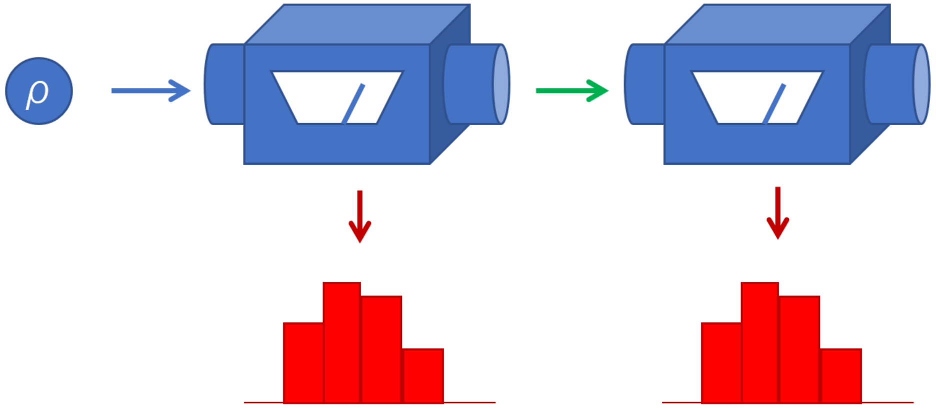
Measurements of the first kind



- An E-instrument $\mathcal{I}$ is a measurement of the first kind if

$$\mathrm{tr}[\mathsf{E}_x \mathcal{I}_{\mathcal{X}}(\rho)] = \mathrm{tr}[\mathsf{E}_x \rho] \,\forall \rho, x$$

Measurements of the first kind



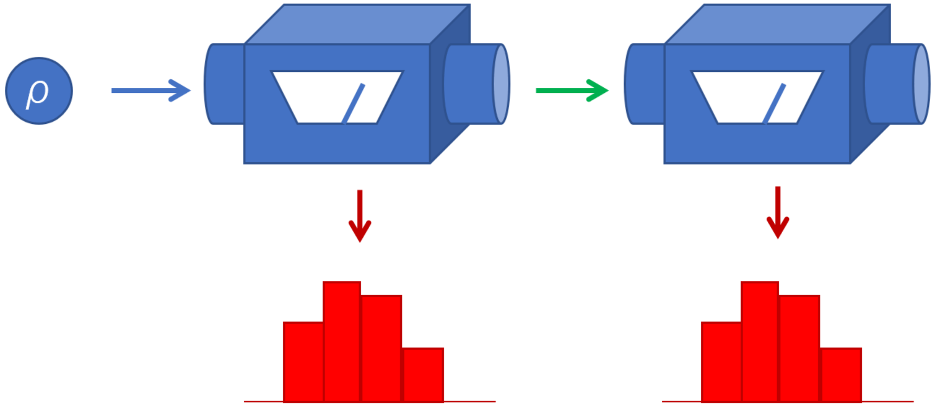
- An E-instrument $\mathcal{I}$ is a measurement of the first kind if

$$\mathrm{tr}[\mathsf{E}_x \mathcal{I}_{\mathcal{X}}(\rho)] = \mathrm{tr}[\mathsf{E}_x \rho]\, \forall \rho, x$$

- Repeatability implies first-kindness, but the converse implication only holds if the observable is sharp

P. J. Lahti, P. Busch, and P. Mittelstaedt, J. Math. Phys. 32, 2770 (1991)

Measurements of the first kind



- An E-instrument $\mathcal{I}$ is a measurement of the first kind if
$$\mathrm{tr}[\mathsf{E}_x \mathcal{I}_{\mathcal{X}}(\rho)] = \mathrm{tr}[\mathsf{E}_x \rho]\, \forall \rho, x$$
- Repeatability implies first-kindness, but the converse implication only holds if the observable is sharp

P. J. Lahti, P. Busch, and P. Mittelstaedt, J. Math. Phys. 32, 2770 (1991)

Theorem. *Under the third law constraint,* E *admits a measurement of the first kind only if none of its effects have eigenvalue 0 or 1, and if it additionally holds that all the effects mutually commute*

Conclusions

Conclusions

- We provided a general formulation of the third law for channels with arbitrary input and output spaces

Conclusions

- We provided a general formulation of the third law for channels with arbitrary input and output spaces

- A measurement constrained by the third law cannot be ideal or repeatable

Conclusions

- We provided a general formulation of the third law for channels with arbitrary input and output spaces
- A measurement constrained by the third law cannot be ideal or repeatable
- But the third law does allow for weakened forms of ideality and repeatability

Conclusions

- We provided a general formulation of the third law for channels with arbitrary input and output spaces
- A measurement constrained by the third law cannot be ideal or repeatable
- But the third law does allow for weakened forms of ideality and repeatability
- Approximately ideal measurements, as well as First-kind measurements, are allowed for “completely unsharp” observables which are indefinite in all states