Classification and coarse graining of quantum cellular automata

Alessandro Bisio

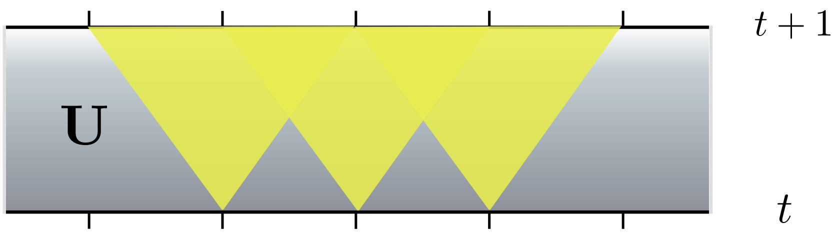
QUANTUM TUT 2024

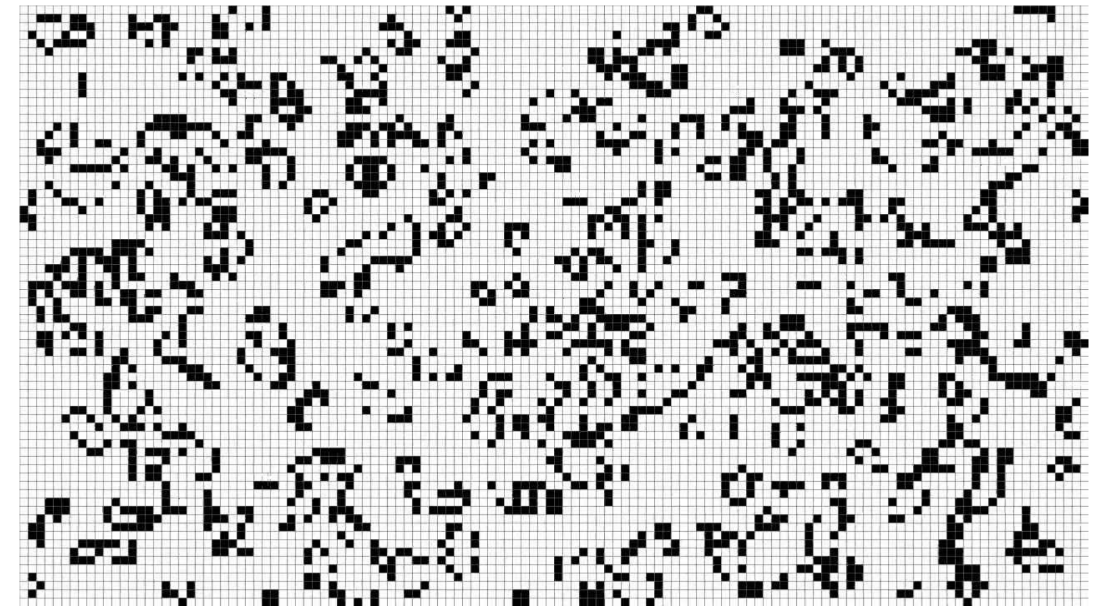
Toyohashi University of Technology, February 22th 2024

Finanziato dall'Unione europea NextGenerationEU
Ministero dell'Università e della Ricerca
Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA
NQSTI National Quantum Science and Technology Institute

Cellular automata

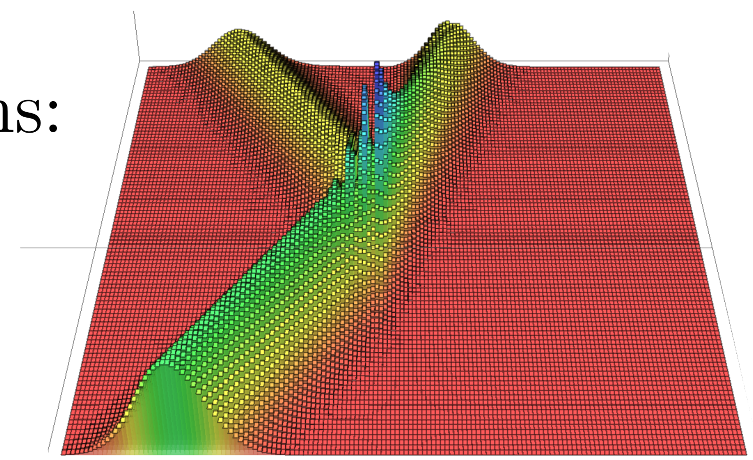
A Cellular automaton is a lattice of systems with some local update rule (we have a "lightcone")

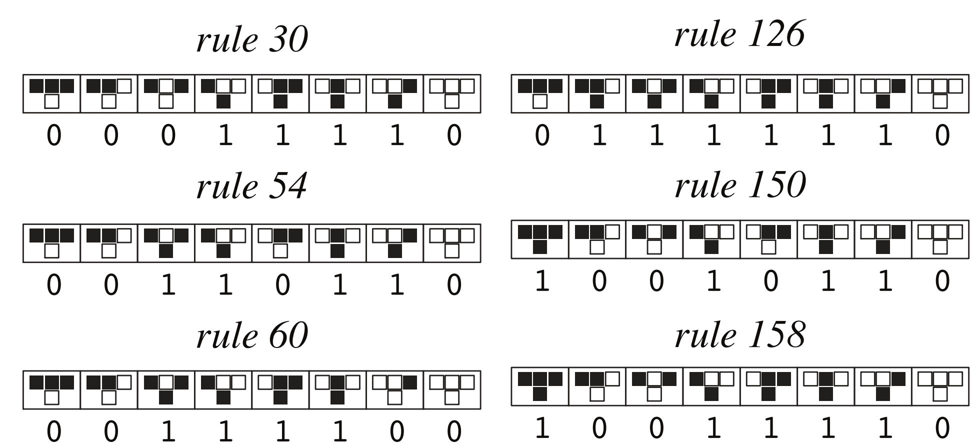


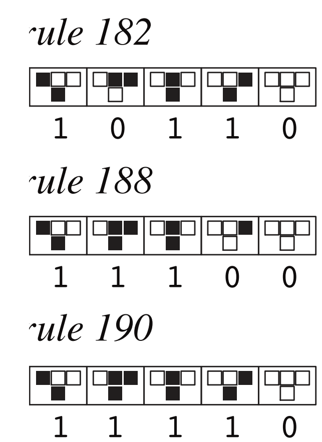


Several applications:

- computation
- simulation
- topological phases of matter







Outline

1) **Quantum Cellular Automata (QCAs): definition and examples**

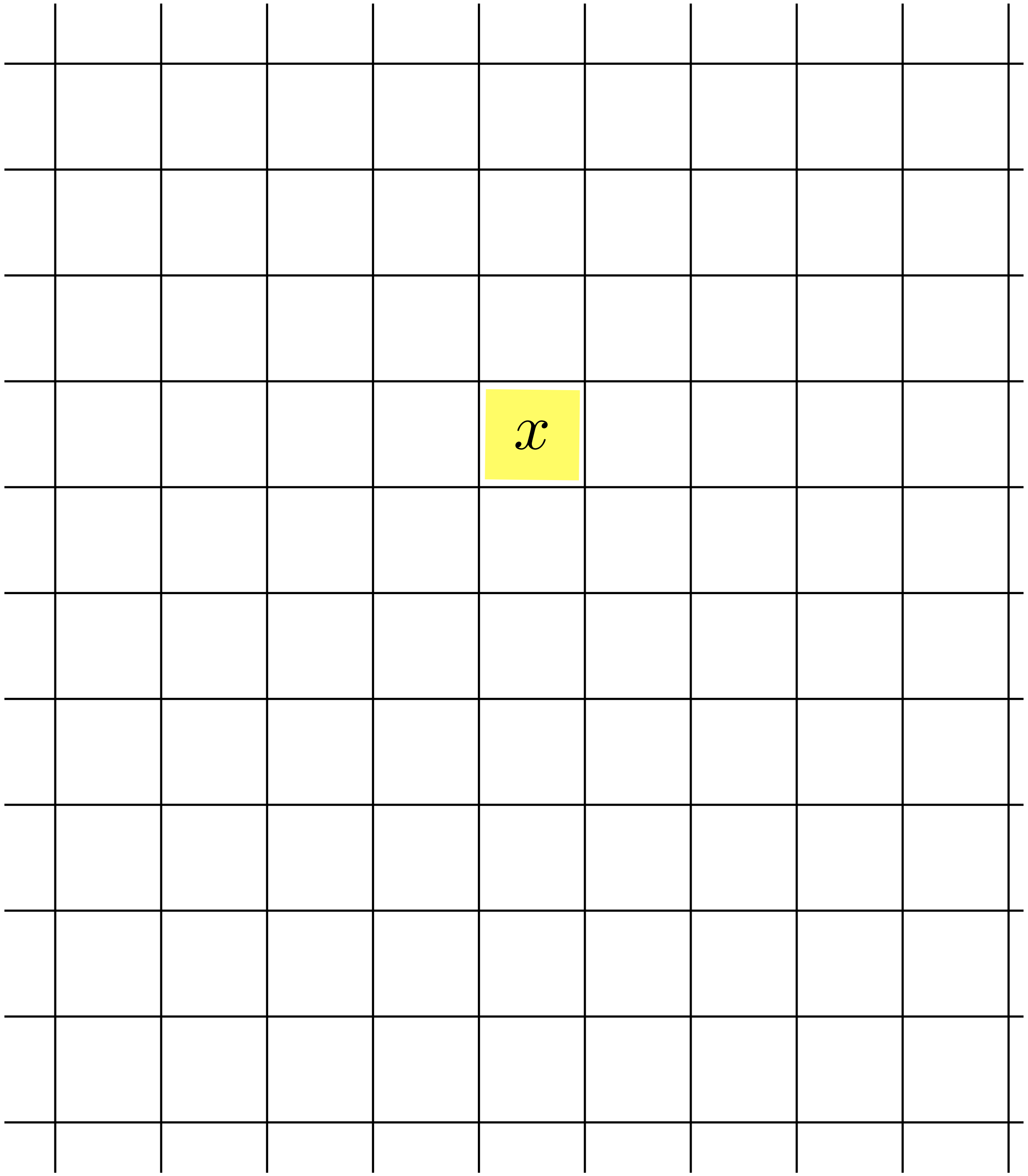
2) **Classification of QCAs**

   - **classifying the rules**
   - **index theory: when a QCA is a circuit**

3) **Coarse graining of QCAs**

QCA: formal definition

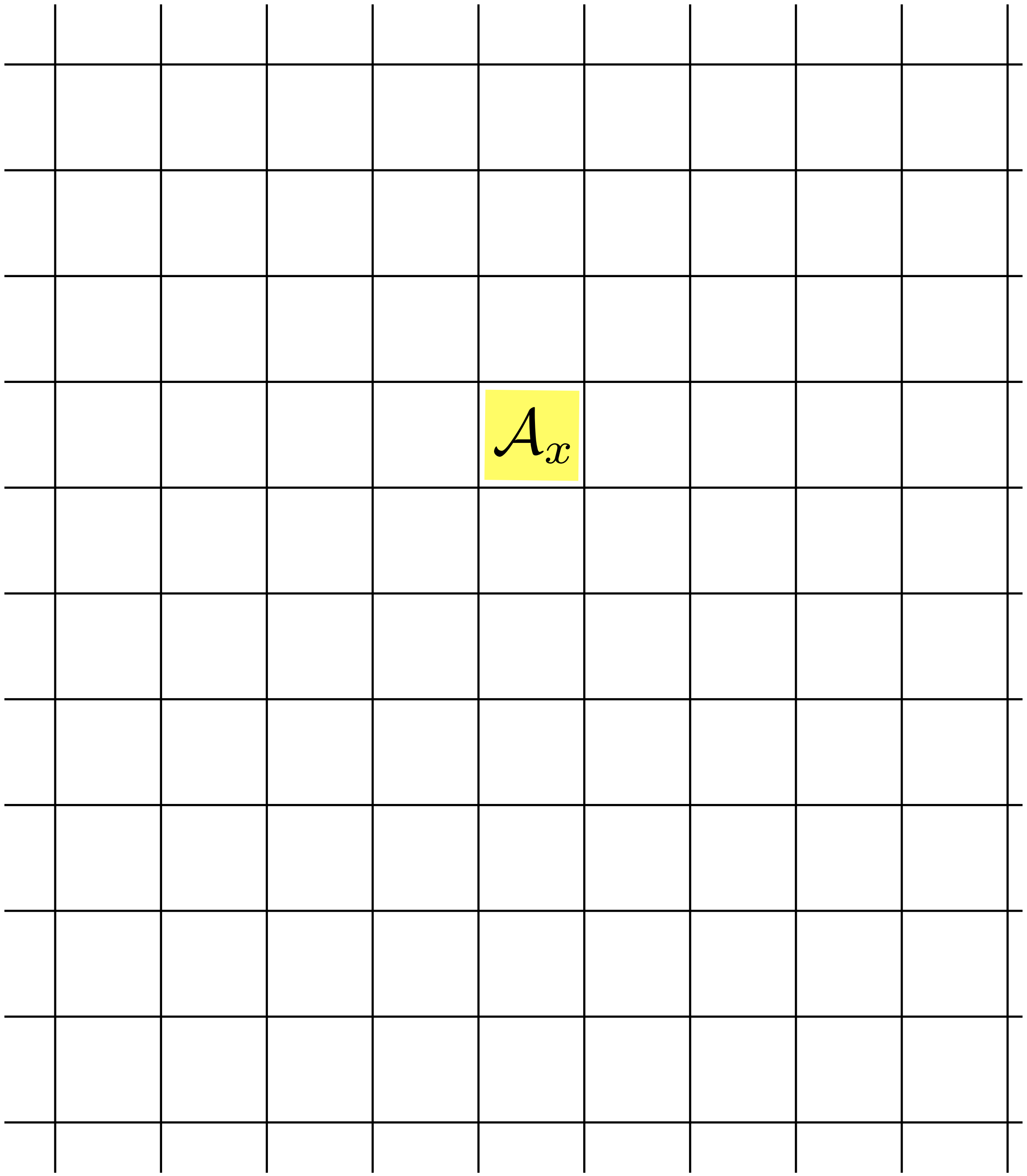
- Lattice of cells, $x \in \mathbb{Z}^s$



QCA: formal definition

- Lattice of cells, $x \in \mathbb{Z}^s$
- $\mathcal{A}_x$ observable algebra at site $\boldsymbol{x}$

  e.g. qubits $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$

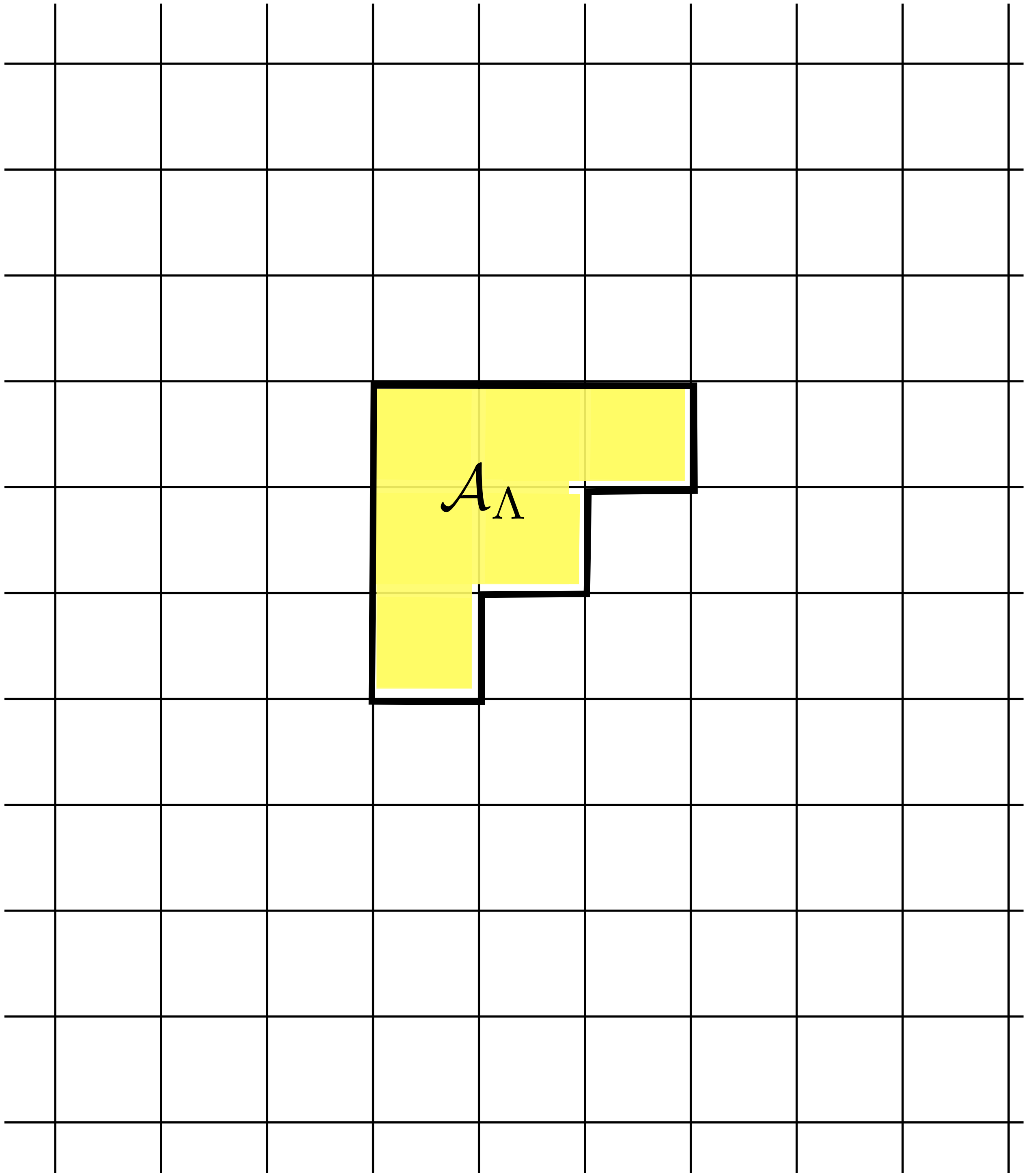


QCA: formal definition

- Lattice of cells, $x \in \mathbb{Z}^s$
- $\mathcal{A}_x$ observable algebra at site $\boldsymbol{x}$

  e.g. qubits $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$
- Local algebras $\mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x$

  $\Lambda \subset \mathbb{Z}^s$ is a **finite** region



QCA: formal definition

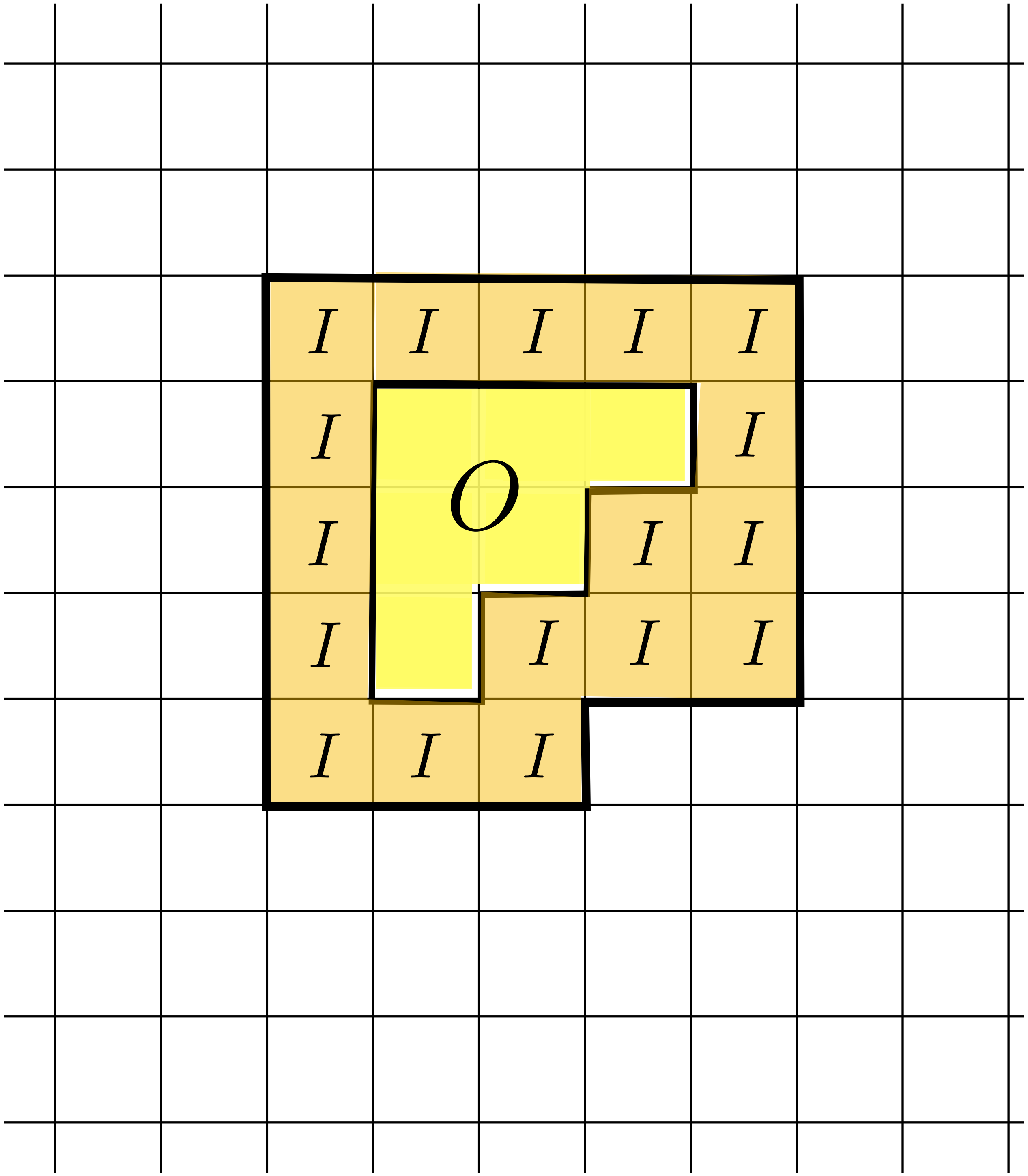
- Lattice of cells, $x \in \mathbb{Z}^s$
- $\mathcal{A}_x$ observable algebra at site $\boldsymbol{x}$

  e.g. qubits $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$
- Local algebras $\mathcal{A}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{A}_x$

  $\Lambda \subset \mathbb{Z}^s$ is a **finite** region

  for $\Lambda \subset \Lambda'$ $\quad O \otimes I_{\Lambda\setminus\Lambda'} \in \mathcal{A}'_\Lambda$

  $\forall O \in \mathcal{A}_\Lambda$.



QCA: formal definition

- Lattice of cells, $x \in \mathbb{Z}^s$
- $\mathcal{A}_x$ observable algebra at site $\boldsymbol{x}$

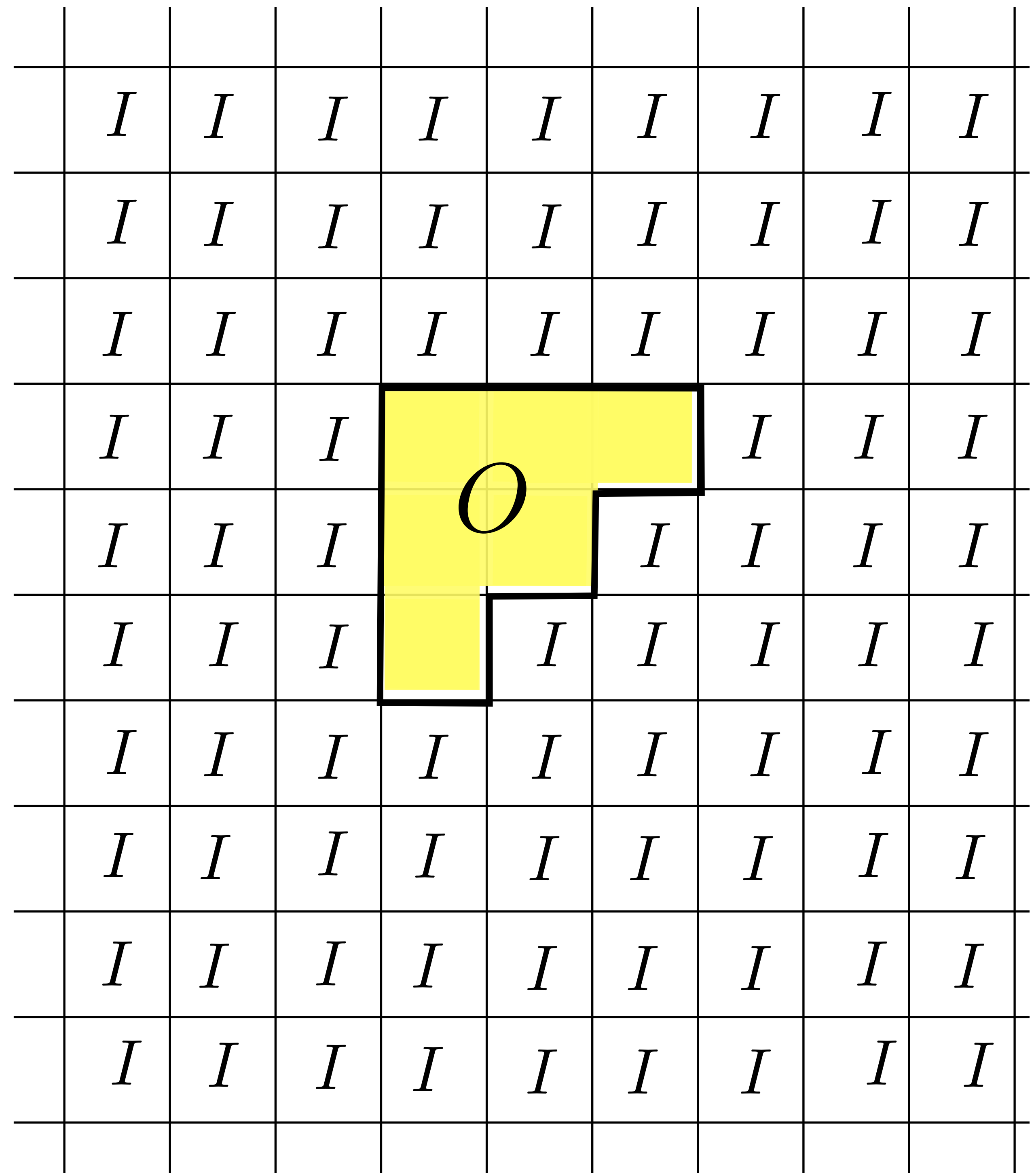
  e.g. qubits $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$
- Local algebras $\mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x$

  $\Lambda \subset \mathbb{Z}^s$ is a **finite** region

  for $\Lambda \subset \Lambda'$ $O \otimes I_{\Lambda \setminus \Lambda'} \in \mathcal{A}'_\Lambda$

  $\forall O \in \mathcal{A}_\Lambda$.
- Quasi-Local algebra

$$\mathcal{A} := \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}$$



QCA: formal definition

A QCA is a local automorphism of the quasi local algebra

QCA: formal definition

A QCA is a local automorphism of the quasi local algebra

$\alpha : \mathcal{A} \to \mathcal{A}$ (injective and surjective)

QCA: formal definition

A QCA is a local automorphism of the quasi local algebra

$\alpha : \mathcal{A} \to \mathcal{A}$ (injective and surjective)

$\alpha(O_1 O_2) = \alpha(O_1)\alpha(O_2)$ (homomorphism)

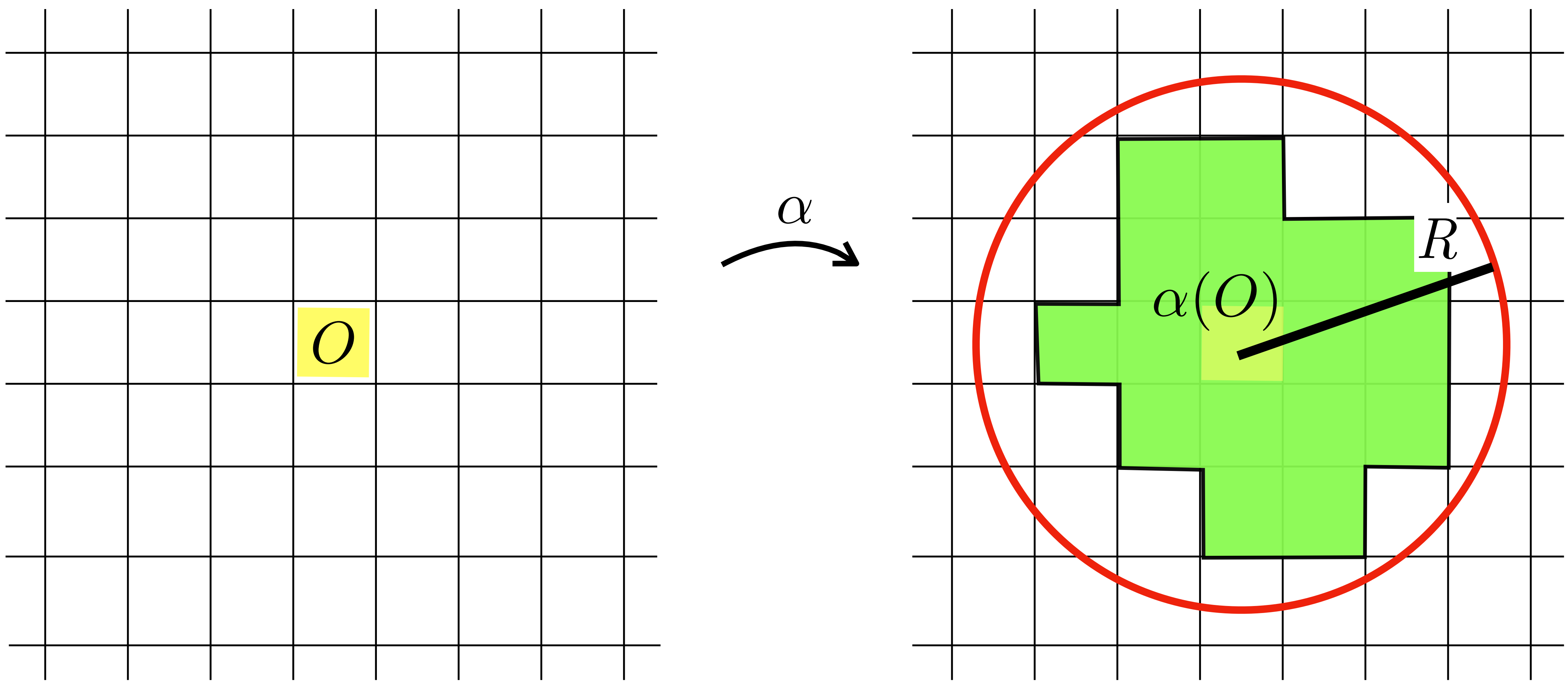
QCA: formal definition

A QCA is a local automorphism of the quasi local algebra

$\alpha : \mathcal{A} \to \mathcal{A}$ (injective and surjective)

$\alpha(O_1 O_2) = \alpha(O_1)\alpha(O_2)$ (homomorphism)

$O \in \mathcal{A}_x \implies \alpha(O) \in \mathcal{A}_{R_x}$ (locality) uniform bound: same R for all x



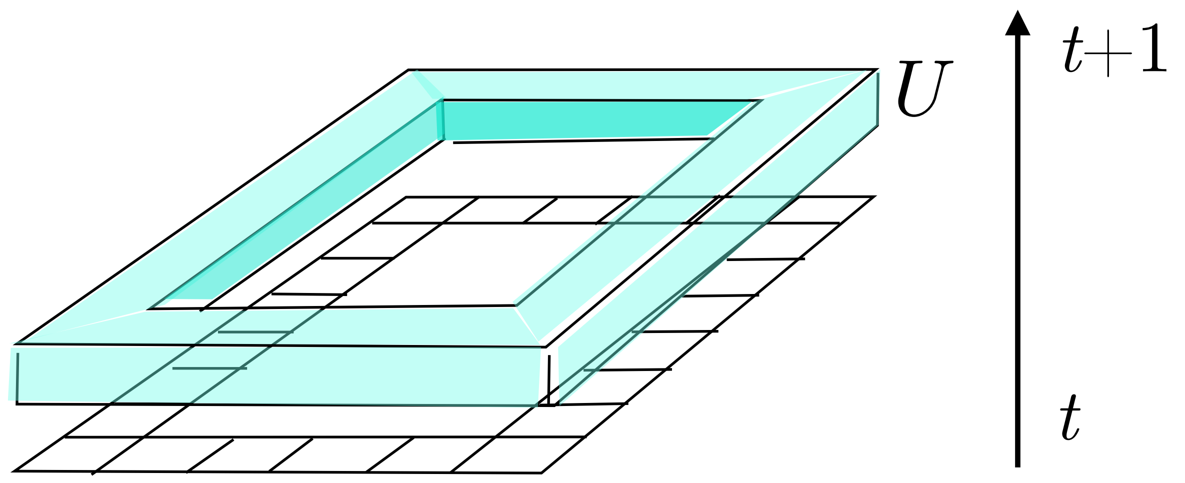
QCA: examples

- Finite lattice, finite dimensional systems at each site

$$\Downarrow$$

$$\alpha(O) = U(O \otimes I)U^\dagger$$

for some unitary matrix U



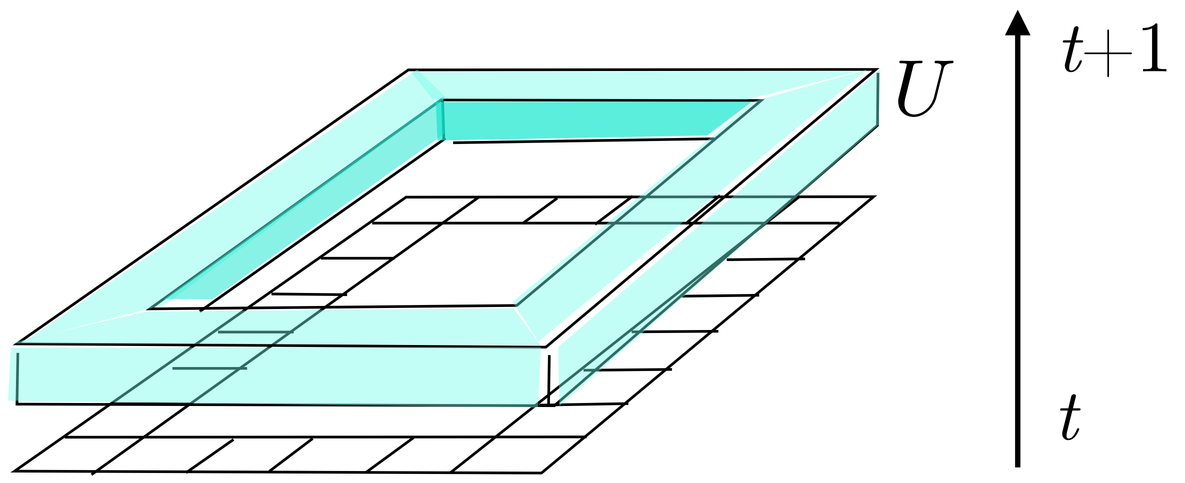
QCA: examples

- Finite lattice, finite dimensional systems at each site

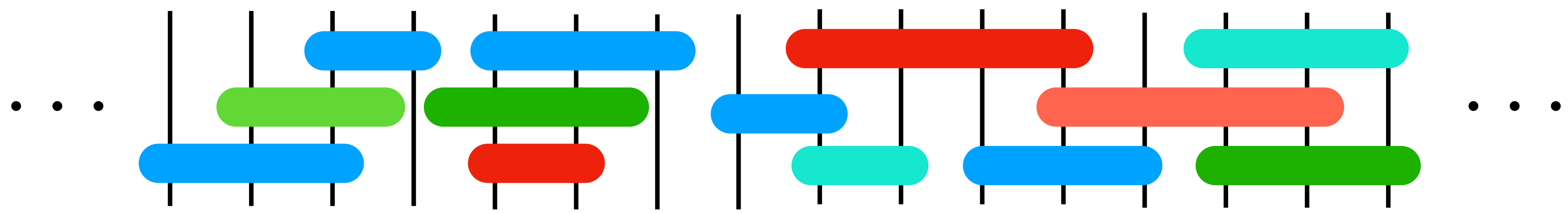
$$\Downarrow$$

$$\alpha(O) = U(O \otimes I)U^\dagger$$

for some unitary matrix U



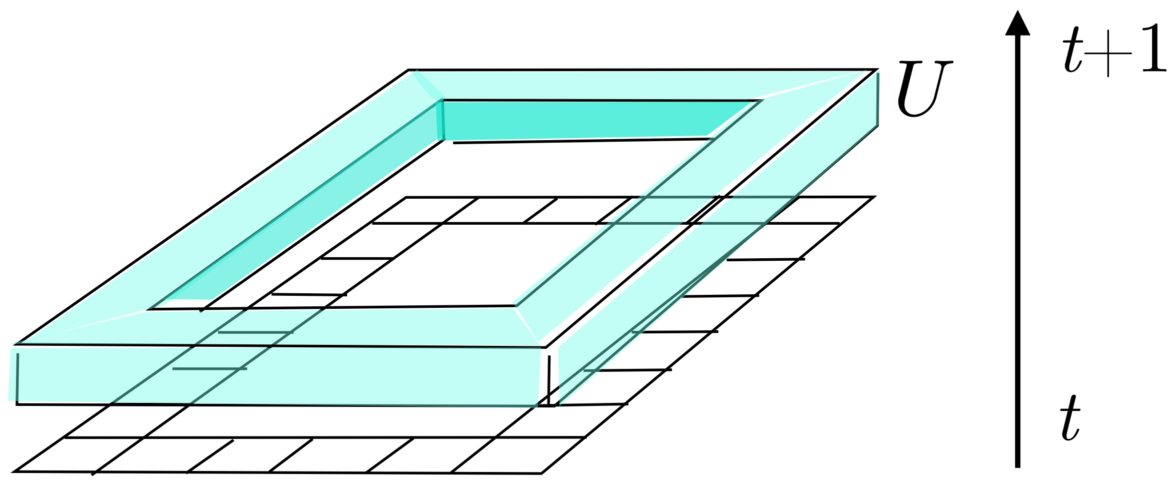
- A quantum circuit is a QCA



BFDQC
bounded
finite depth
quantum circuit

QCA: examples

- Finite lattice, finite dimensional systems at each site

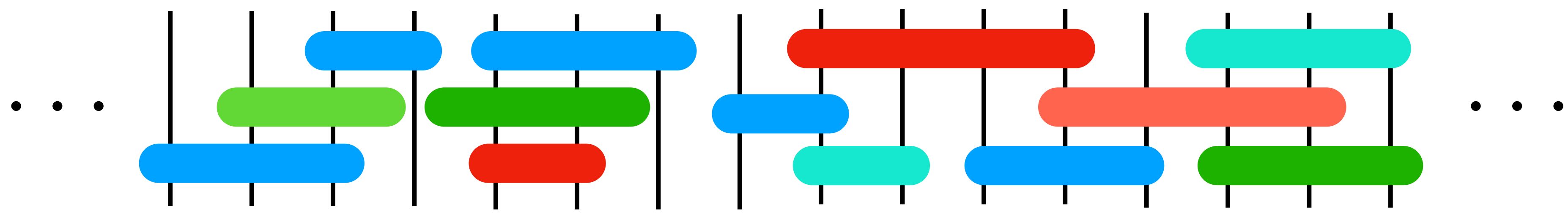


$$\Downarrow$$

$$\alpha(O) = U(O \otimes I)U^\dagger$$

for some unitary matrix U

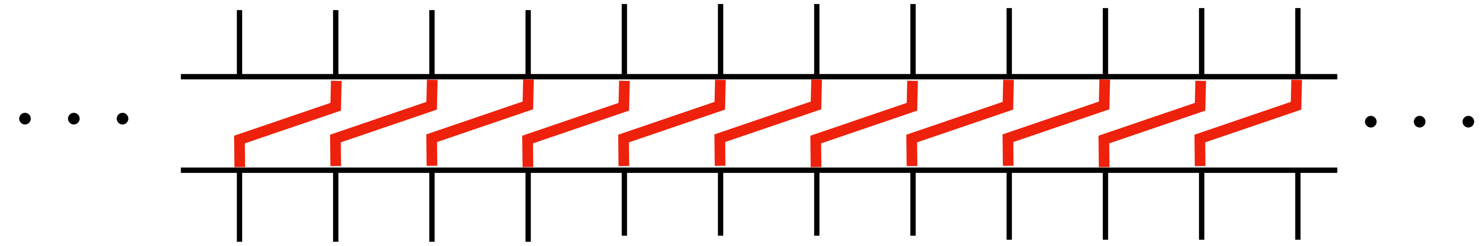
- A quantum circuit is a QCA



BFDQC
bounded
finite depth
quantum circuit

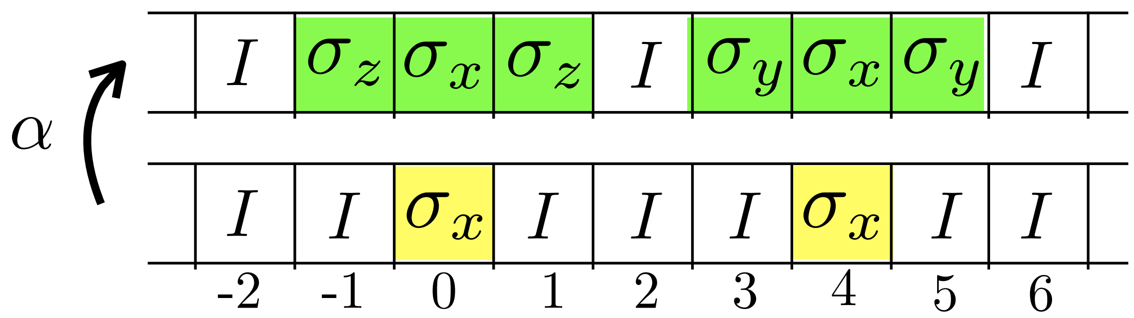
- But not every QCA is a circuit!

the shift
is NOT a circuit



QCA: examples

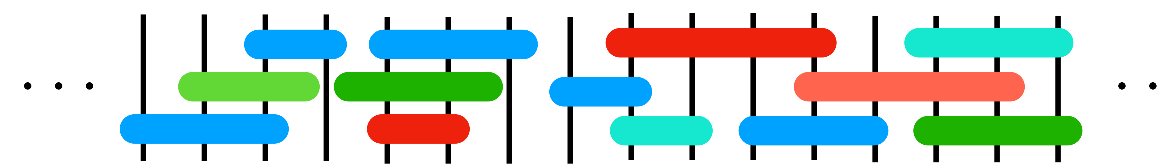
- The QCA may give an update rule which depends on the site



$$\alpha|_0(\sigma_x) = \sigma_z \otimes \sigma_x \otimes \sigma_z$$

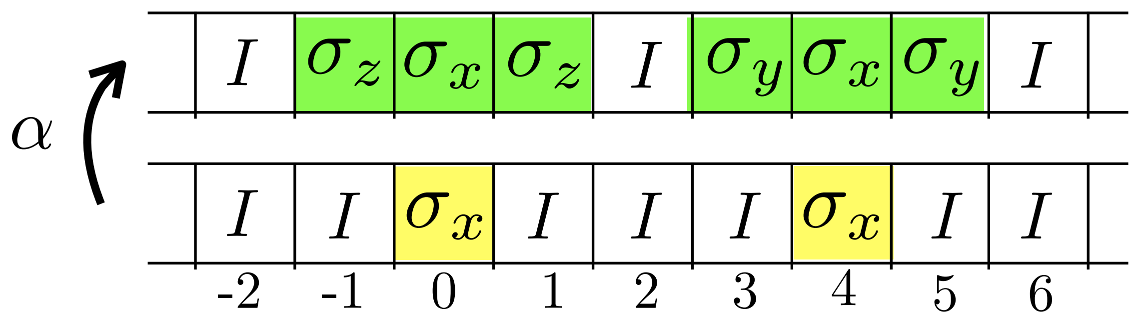
$$\alpha|_4(\sigma_x) = \sigma_y \otimes \sigma_x \otimes \sigma_y$$

In this case, we say that the QCA is NOT translation invariant



QCA: examples

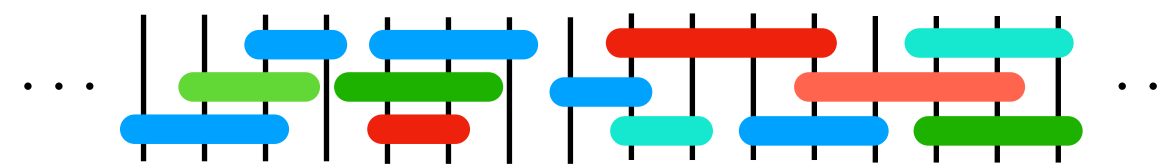
- The QCA may give an update rule which depends on the site



$$\alpha|_0(\sigma_x) = \sigma_z \otimes \sigma_x \otimes \sigma_z$$

$$\alpha|_4(\sigma_x) = \sigma_y \otimes \sigma_x \otimes \sigma_y$$

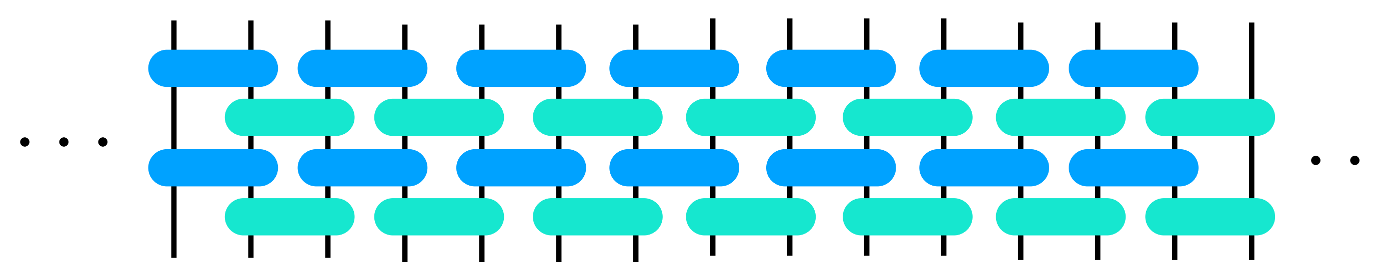
In this case, we say that the QCA is NOT translation invariant



- If the QCA acts in the same way everywhere on the lattice, we say that the QCA is translation invariant

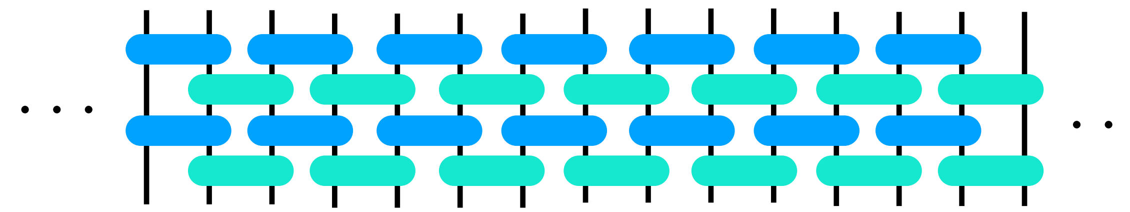
$$\alpha|_x = \alpha|_0 \quad \forall x$$

"the" local rule



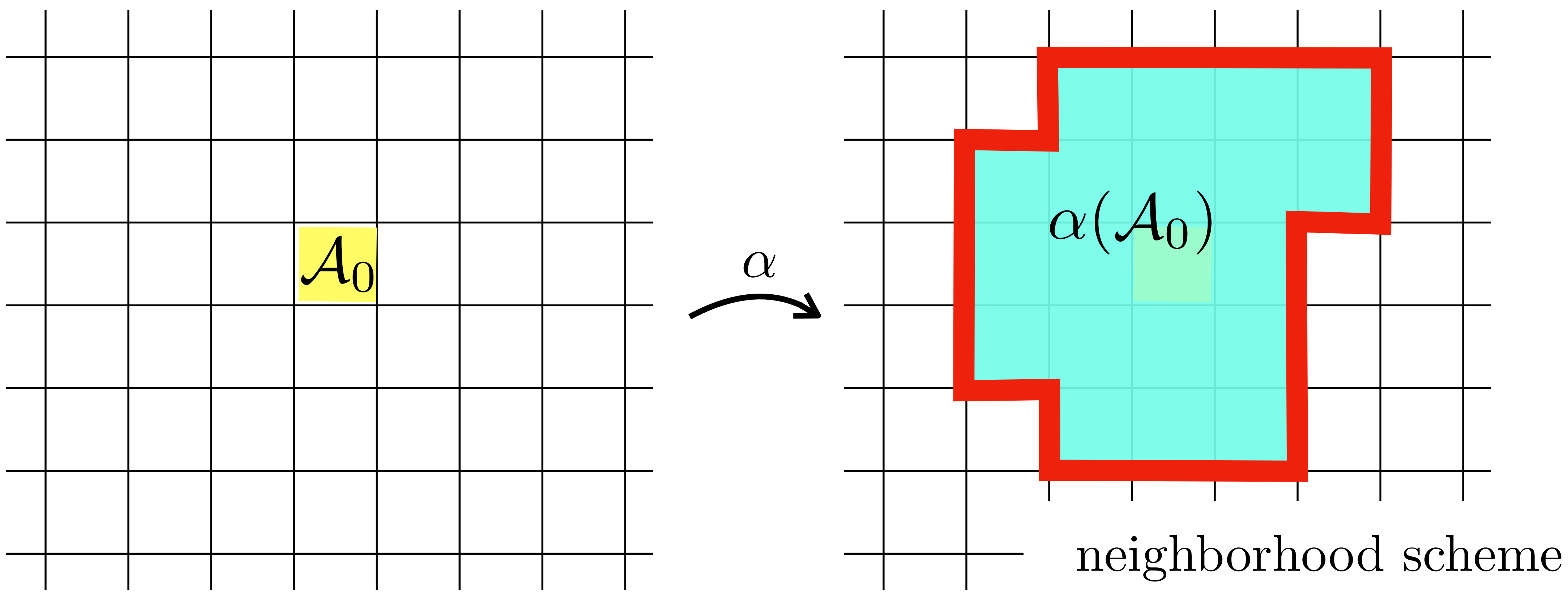
QCA: examples

- If the QCA acts in the same way everywhere on the lattice, we say that the QCA is translation invariant



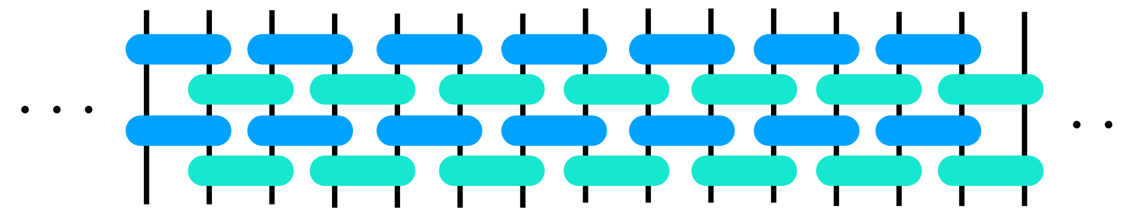
$$\alpha|_x = \alpha|_0 \quad \forall x$$

"the" local rule



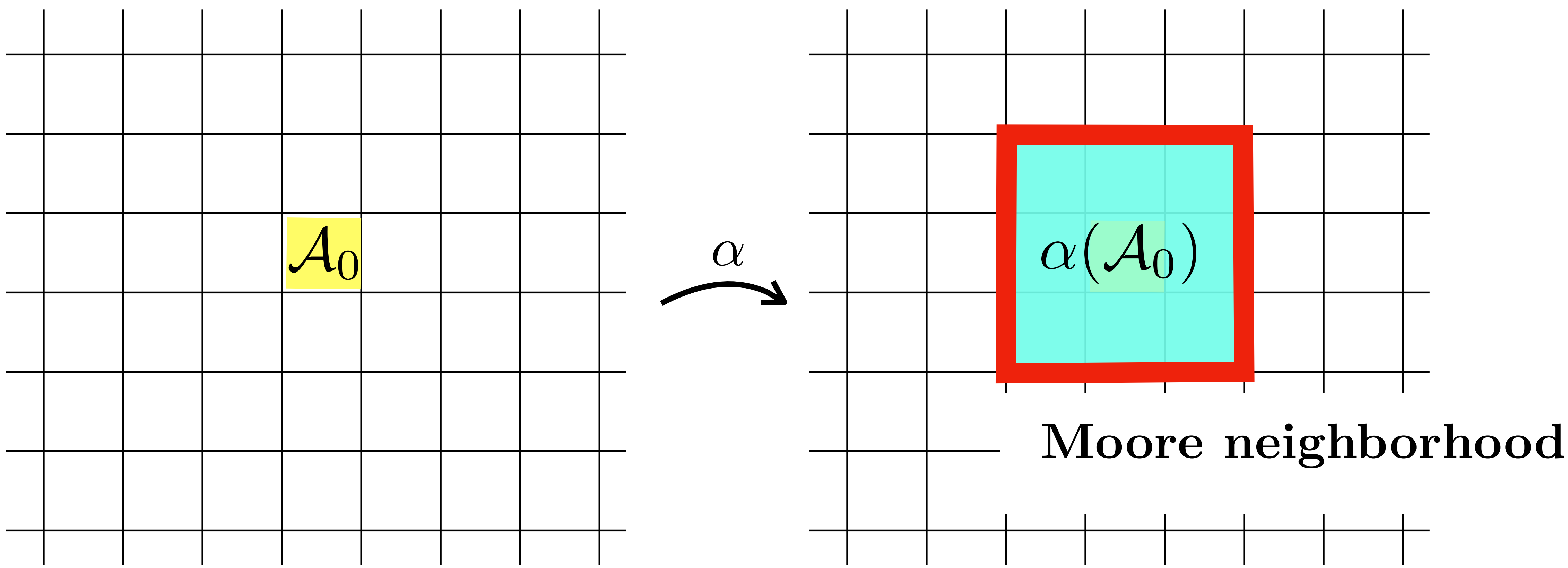
QCA: examples

- If the QCA acts in the same way everywhere on the lattice, we say that the QCA is translation invariant



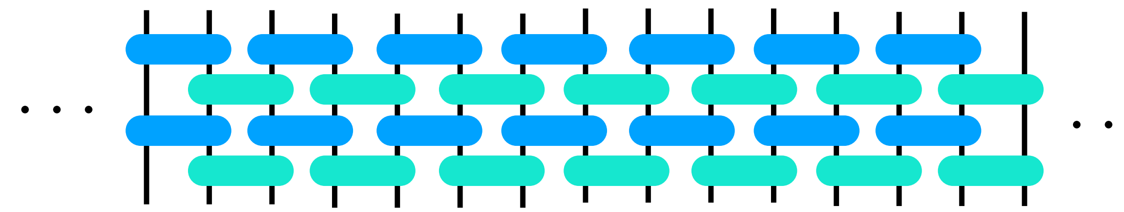
$$\alpha|_x = \alpha|_0 \quad \forall x$$

“the” local rule



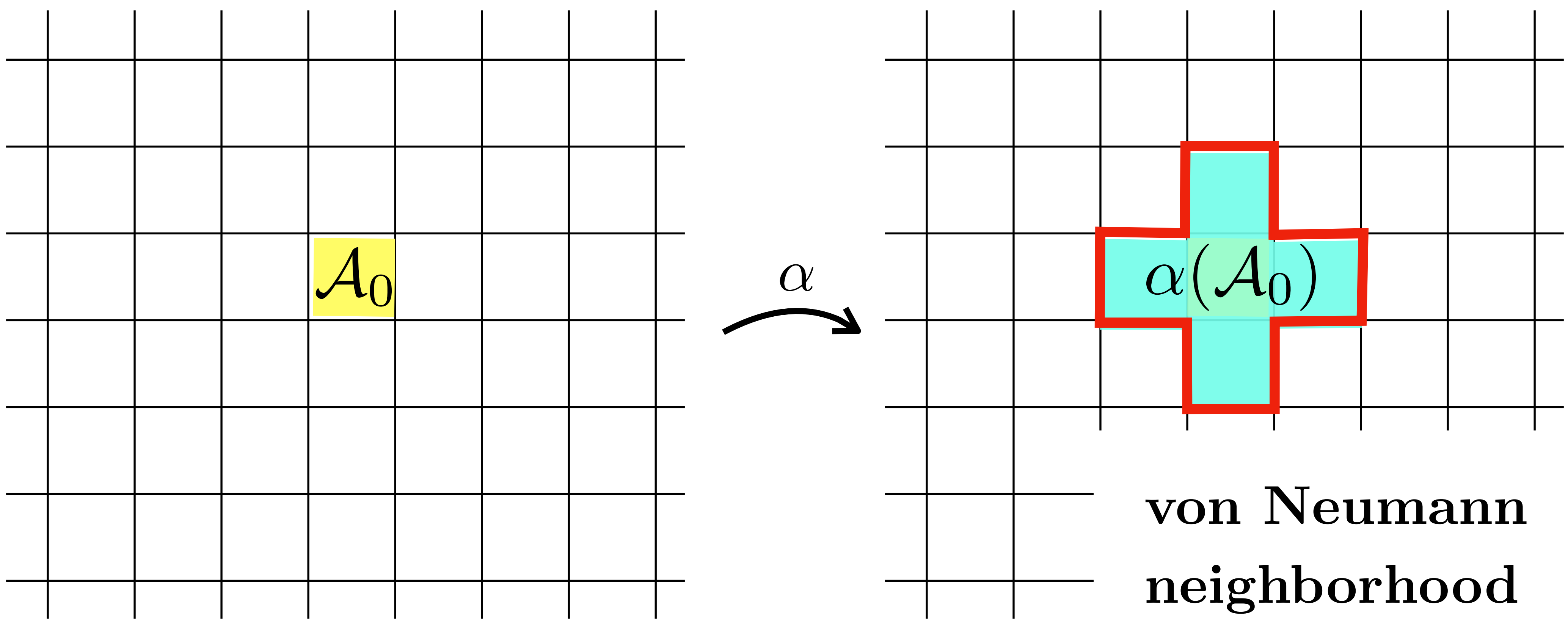
QCA: examples

- If the QCA acts in the same way everywhere on the lattice, we say that the QCA is translation invariant



$$\alpha|_x = \alpha|_0 \quad \forall x$$

"the" local rule



Classification of QCA

How many QCAs are there?

Classification of QCA

How many QCAs are there?

assuming:

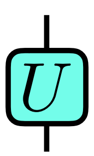
- **Translation invariance**
- **qubit cells** $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$
- **von Neumann neighborhood**

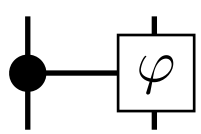
Classification of QCA

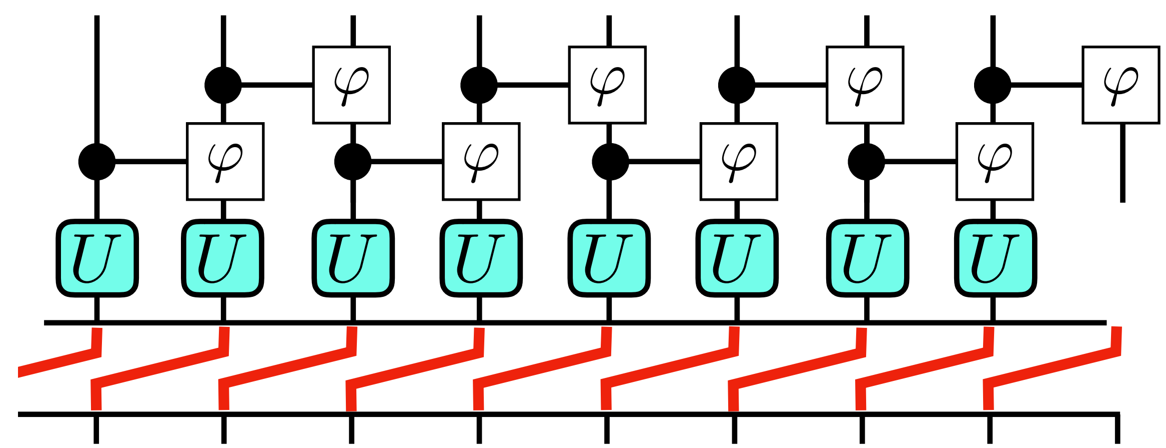
How many QCAs are there?

assuming:
- **Translation invariance**
- **qubit cells** $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$
- **von Neumann neighborhood**

1 dimensional qubit QCA


arbitrary unitary gate

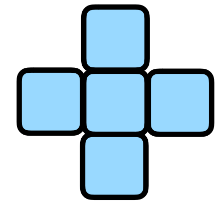

c-phase gate



B. Schumacher, R.F. Werner
e-print arXiv:0405174.

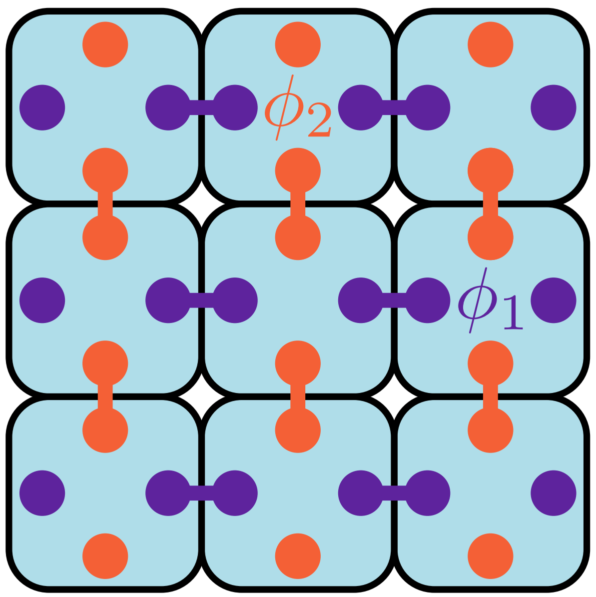
Classification of QCA

2 dimensional qubit QCA

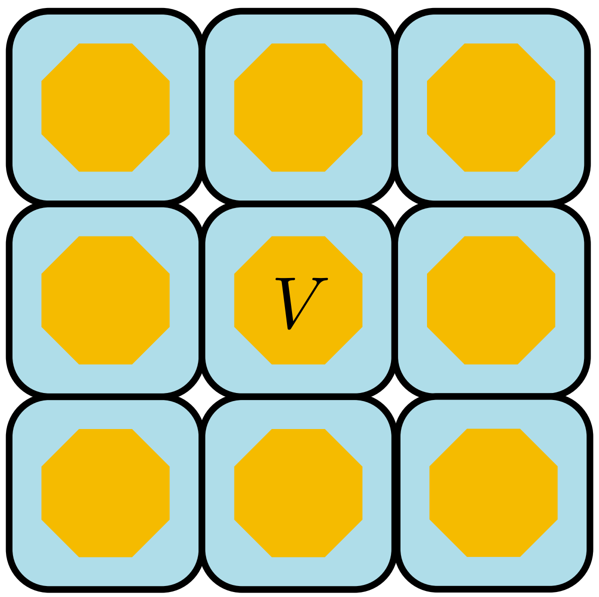


von Neumann
neighborhood in 2D

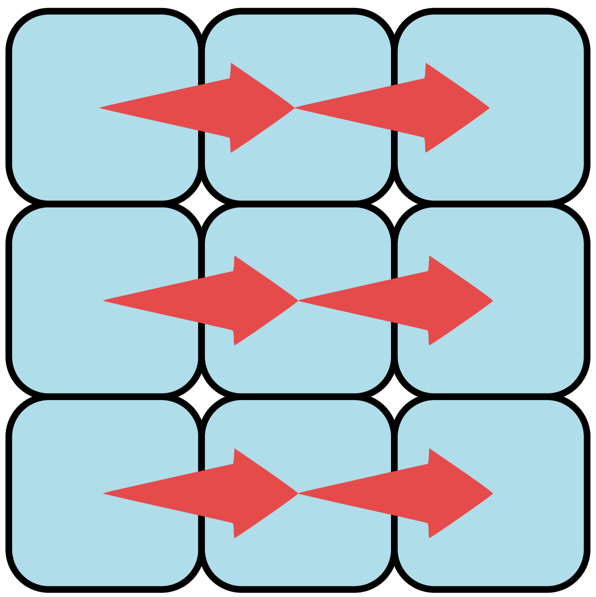
c-phase gates



local unitary gate



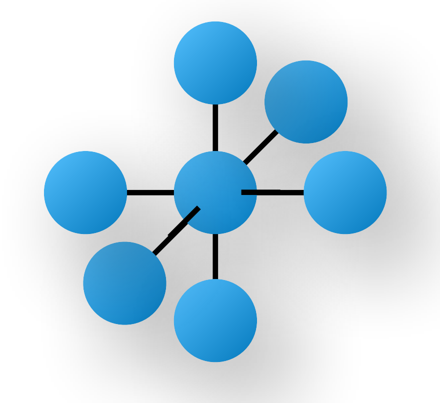
shift



AB, A. Pizzamiglio, P. Perinotti
in preparation

N dimensional qubit QCA

(as in the 2D case)



von Neumann
neighborhood in 3D

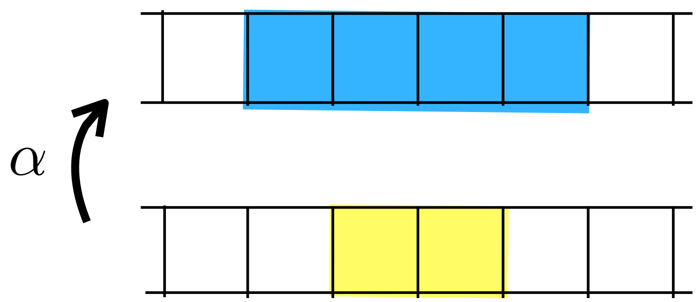
Classification of QCA: index theory

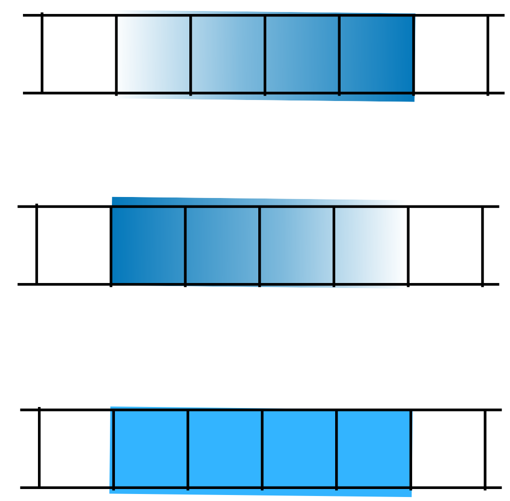
When a QCA is quantum circuit?

Classification of QCA: index theory

When a QCA is quantum circuit?

1 dimensional QCA





$i(\alpha) > 0$
$i(\alpha) < 0$
Not a circuit

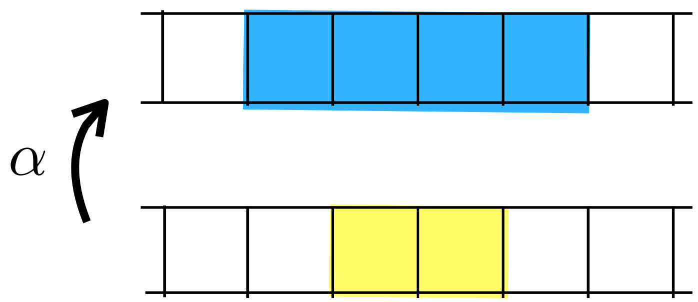
$i(\alpha) = 0$ α is a circuit

D. Gross, V. Nesme, H. Vogts, R.F. Werner
CMP 310, 419 (2012)

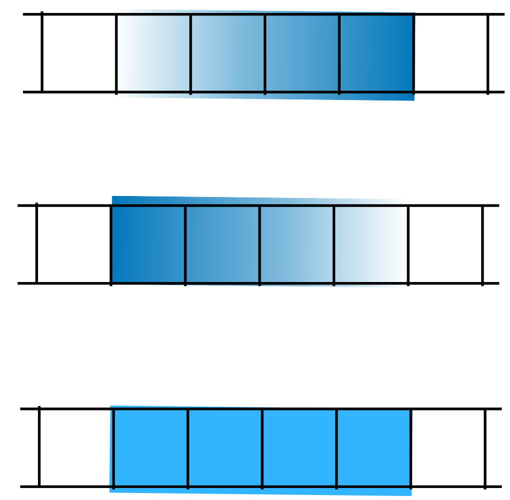
Classification of QCA: index theory

When a QCA is quantum circuit?

1 dimensional QCA



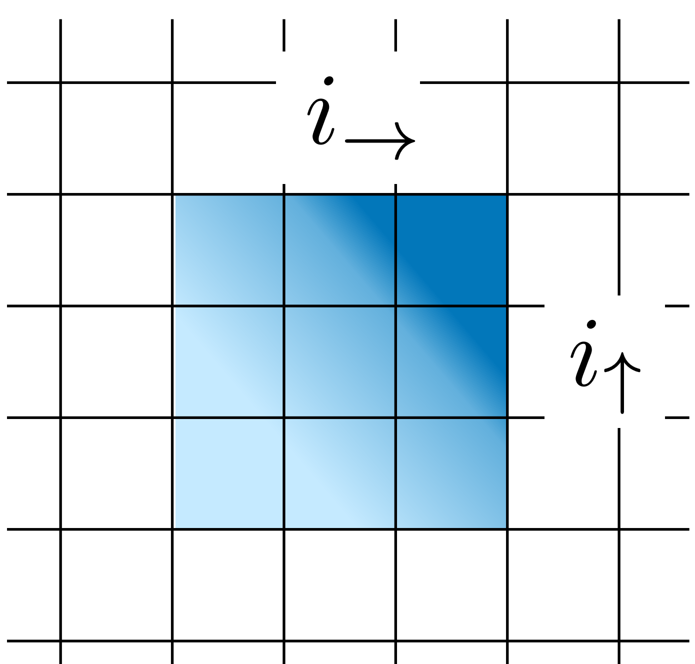
D. Gross, V. Nesme, H. Vogts, R.F. Werner
CMP 310, 419 (2012)



$i(\alpha) > 0$, $i(\alpha) < 0$: Not a circuit

$i(\alpha) = 0$: α is a circuit

2 dimensional QCA (translation invariant)



$$\alpha \text{ is a circuit} \Updownarrow i_{\rightarrow}(\alpha) = i_{\downarrow}(\alpha) = 0$$

M. Freedman, M. Hastings
CMP 376, 11071 (2020)

AB, P. Perinotti, A. Pizzamiglio
in preparation

Coarse graining of CA

B is a coarse graining of A

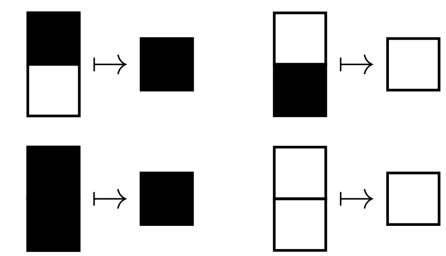
$$P \circ f_A^T = f_B \circ P$$

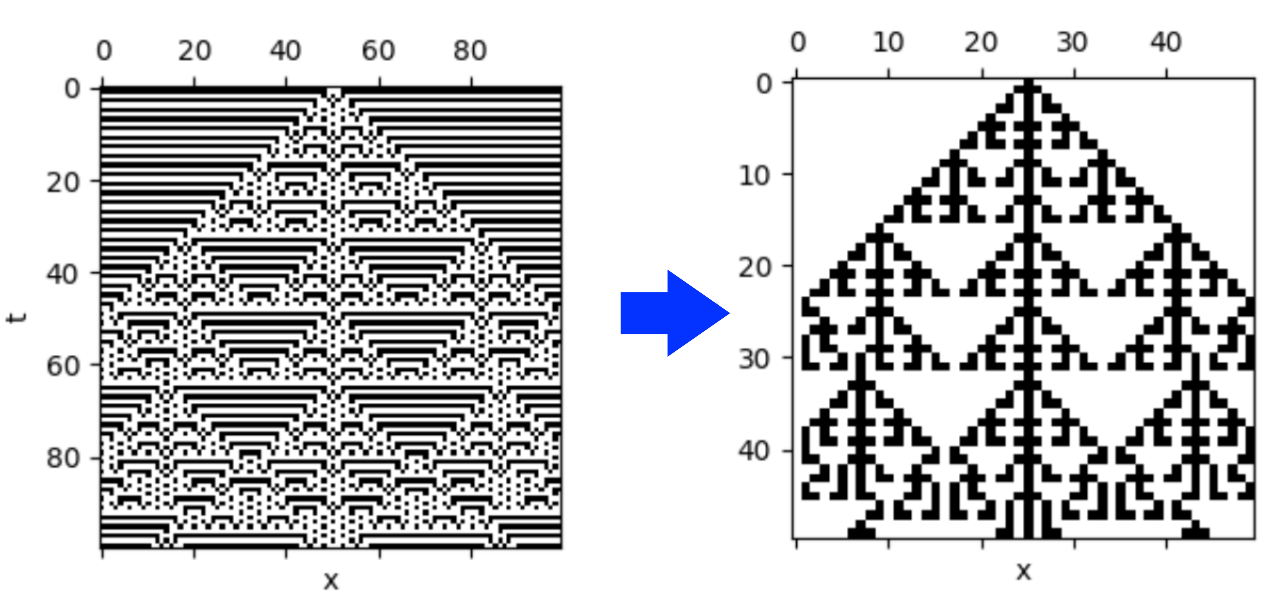
update rule

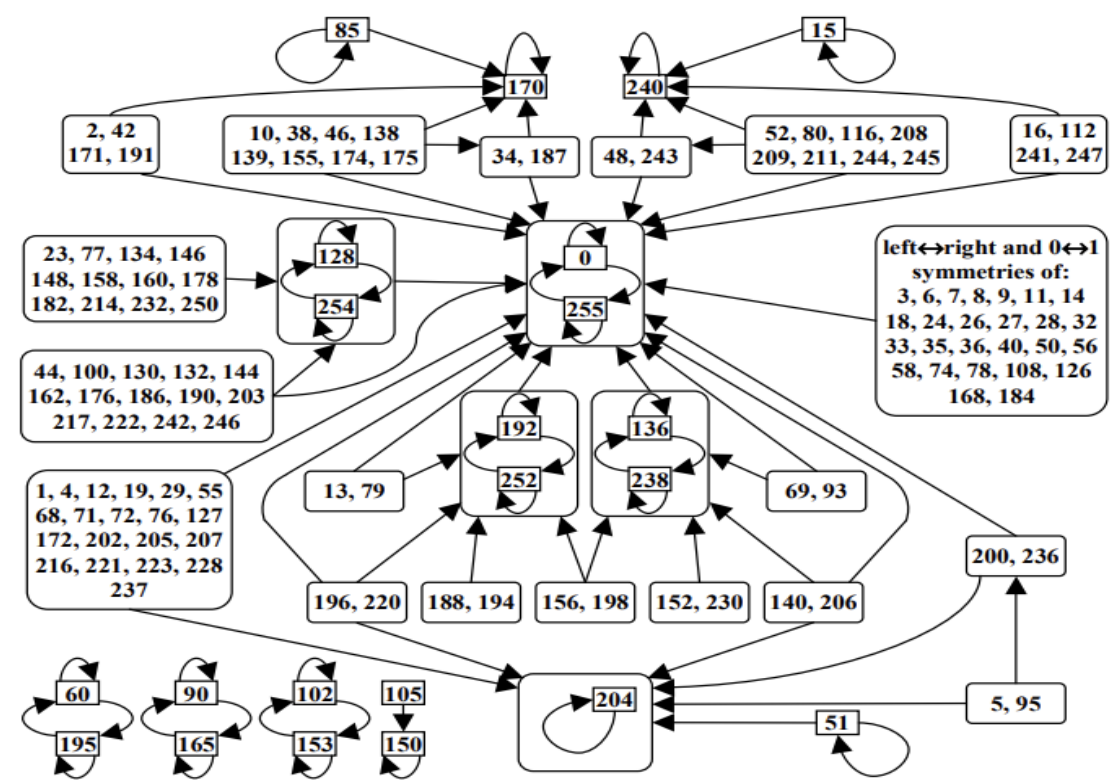
N cells → 1 cell

T steps → 1 step

P :





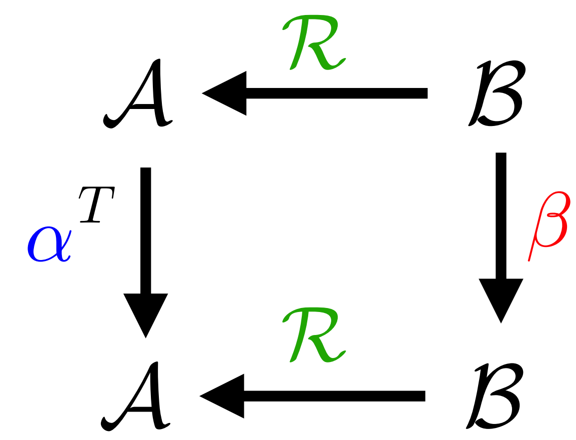


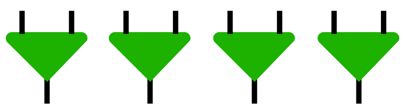
N. Israeli, N. Goldenfeld PRE 73, 026203 (2006)

Coarse graining of translation invariant QCA

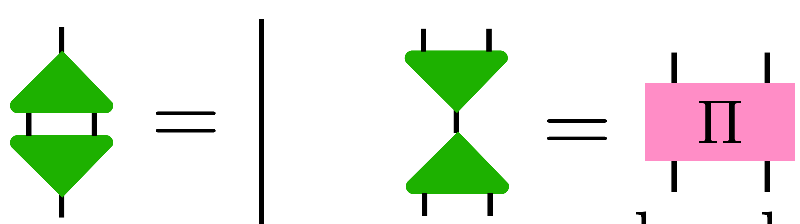
β **is a coarse graining of** α

$$\alpha^T \circ \mathcal{R} = \mathcal{R} \circ \beta$$

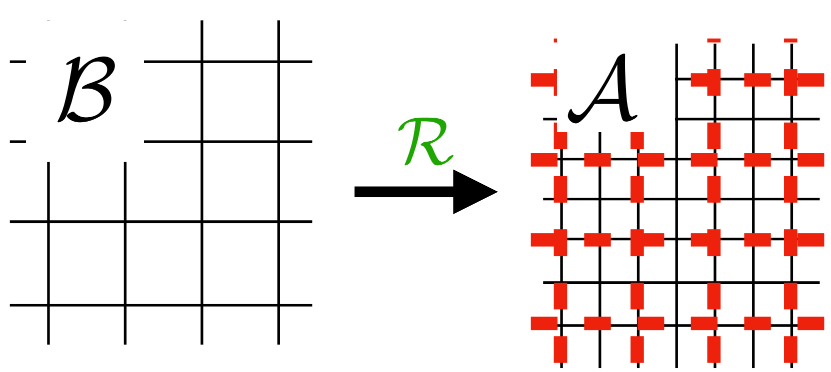


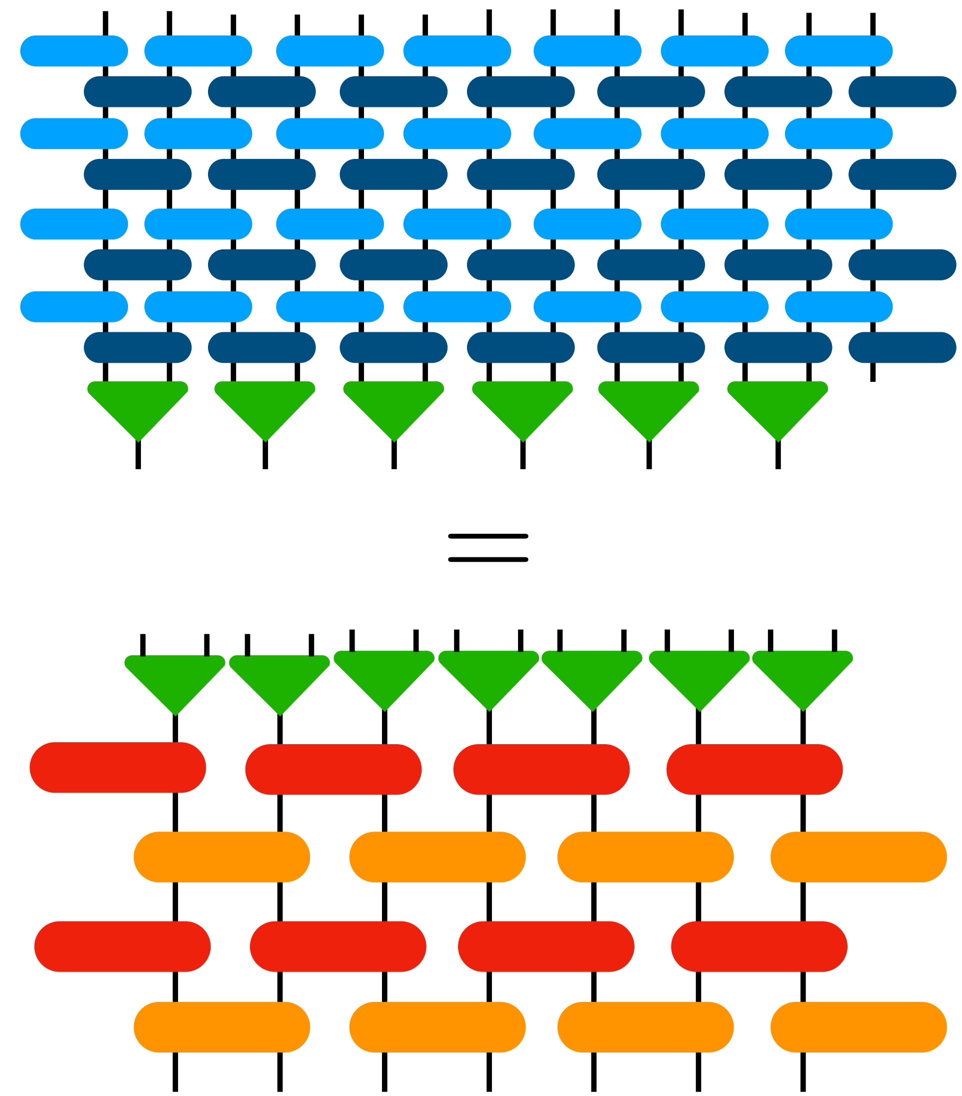
$\mathcal{R} = \cdots$  $\cdots$

is an isometric homomorphism



local projector





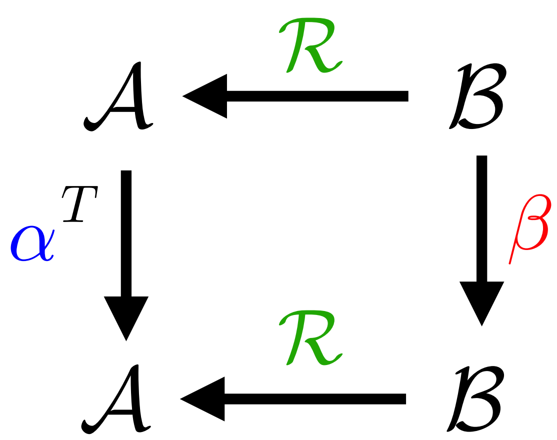
AB, P. Perinotti, L, Trezzini, in preparation

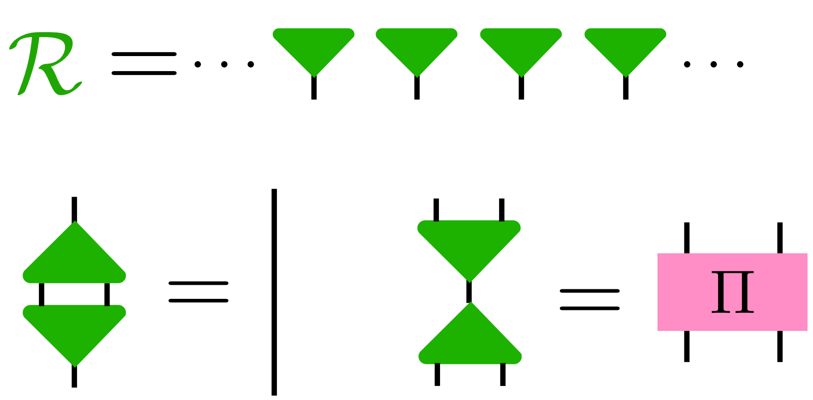
Coarse graining of QCA

When does a QCA α admit a coarse graining β ?

Coarse graining of QCA

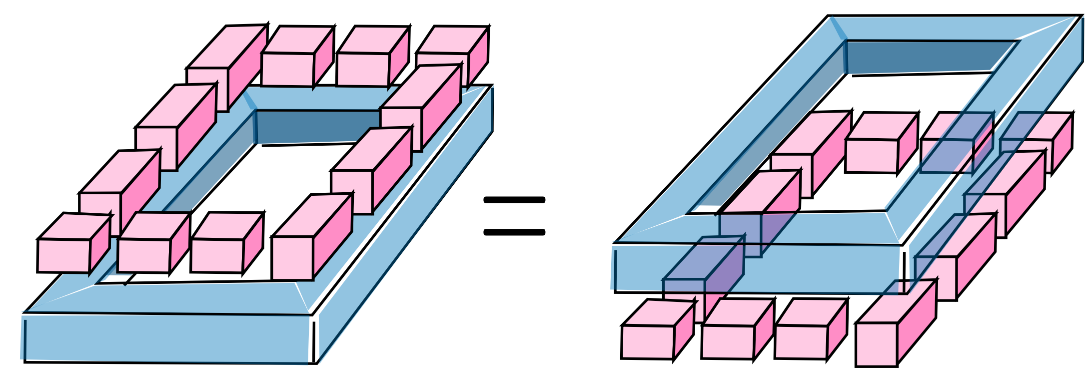
When does a QCA α admit a coarse graining β ?

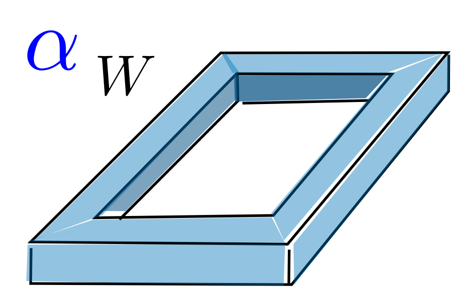




$\Longleftrightarrow$

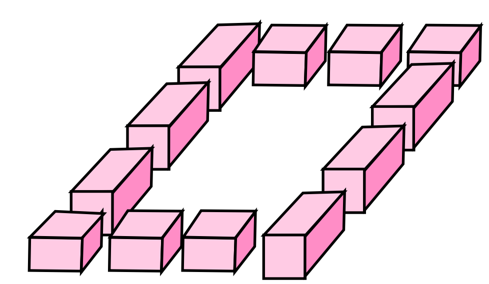
$$\alpha^T_W(\Pi_W) = \Pi_W$$





QCA on an arbitrary finite lattice

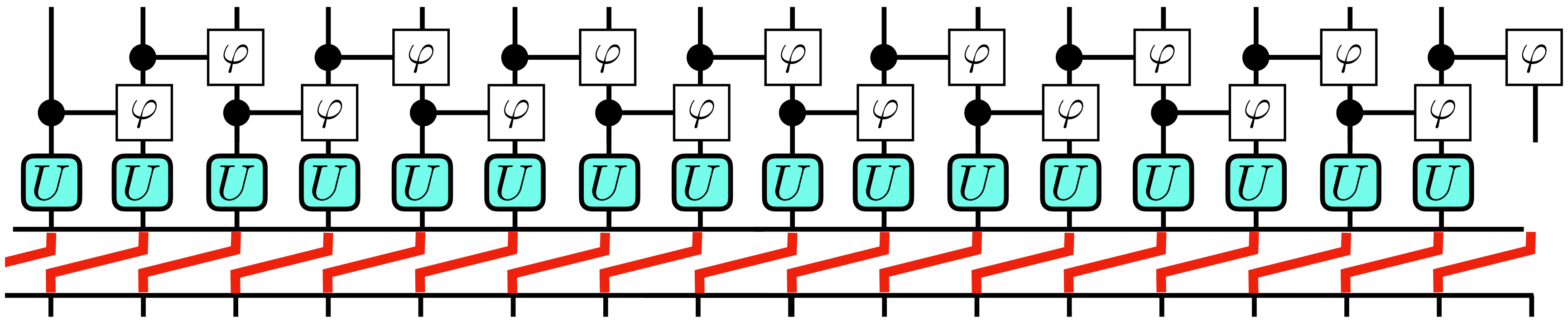
$$\Pi_W := \bigotimes_{i \in W} \Pi_i$$



AB, P. Perinotti, L, Trezzini, in preparation

Coarse graining of qubit 1D QCA (2 steps, 2 cells)

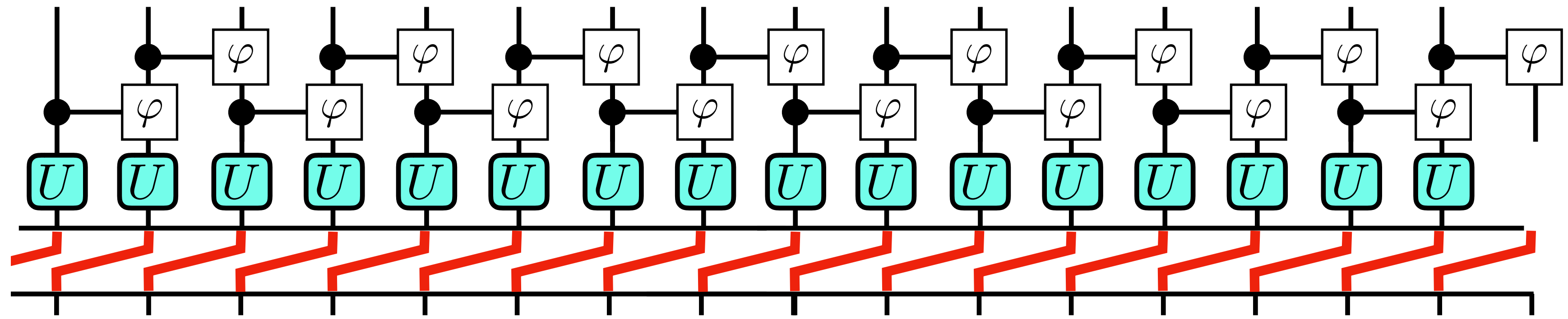
The qubit 1D QCA are classified



AB, P. Perinotti, L, Trezzini, in preparation

Coarse graining of qubit 1D QCA (2 steps, 2 cells)

The qubit 1D QCA are classified

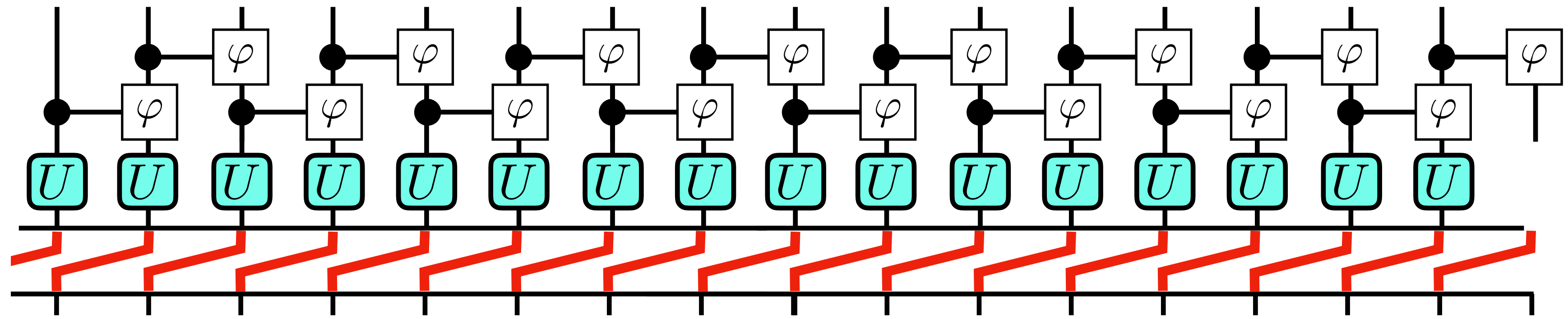


Results: • the index cannot change

AB, P. Perinotti, L, Trezzini, in preparation

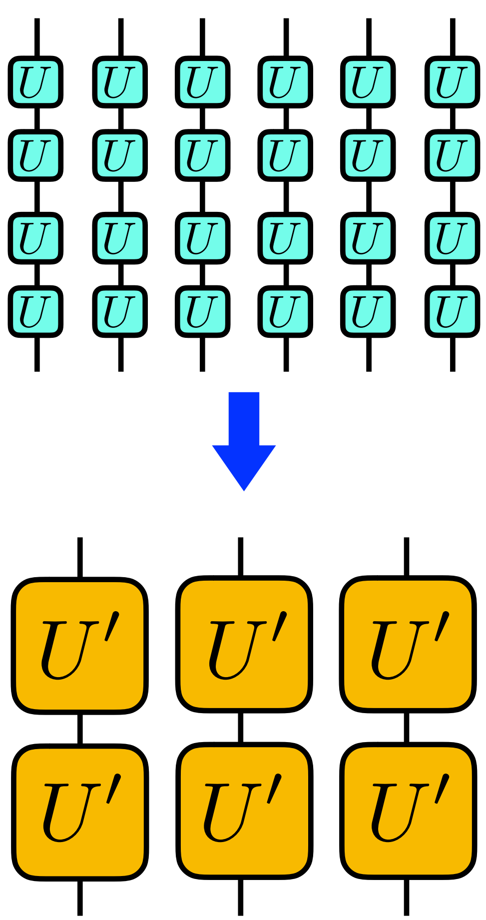
Coarse graining of qubit 1D QCA (2 steps, 2 cells)

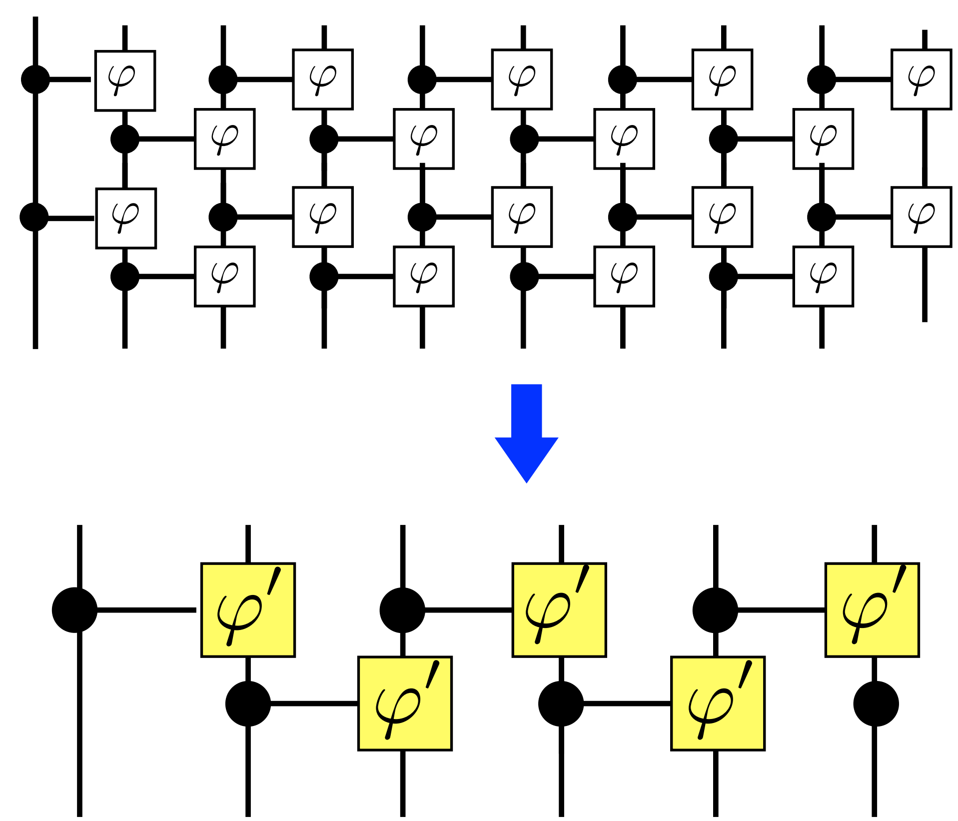
The qubit 1D QCA are classified



Results:

- the index cannot change
- the only 1D qubit QCA that admit a 2 cells coarse graining are “trivial” (no propagation of information)





AB, P. Perinotti, L, Trezzini, in preparation

Summary

Which local rules give rise to a QCA?

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann neighborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell whether a QCA is a circuit or not?

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell whether a QCA is a circuit or not?

Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell whether a QCA is a circuit or not?

Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

What does it mean that α is a coarse graining of β?

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

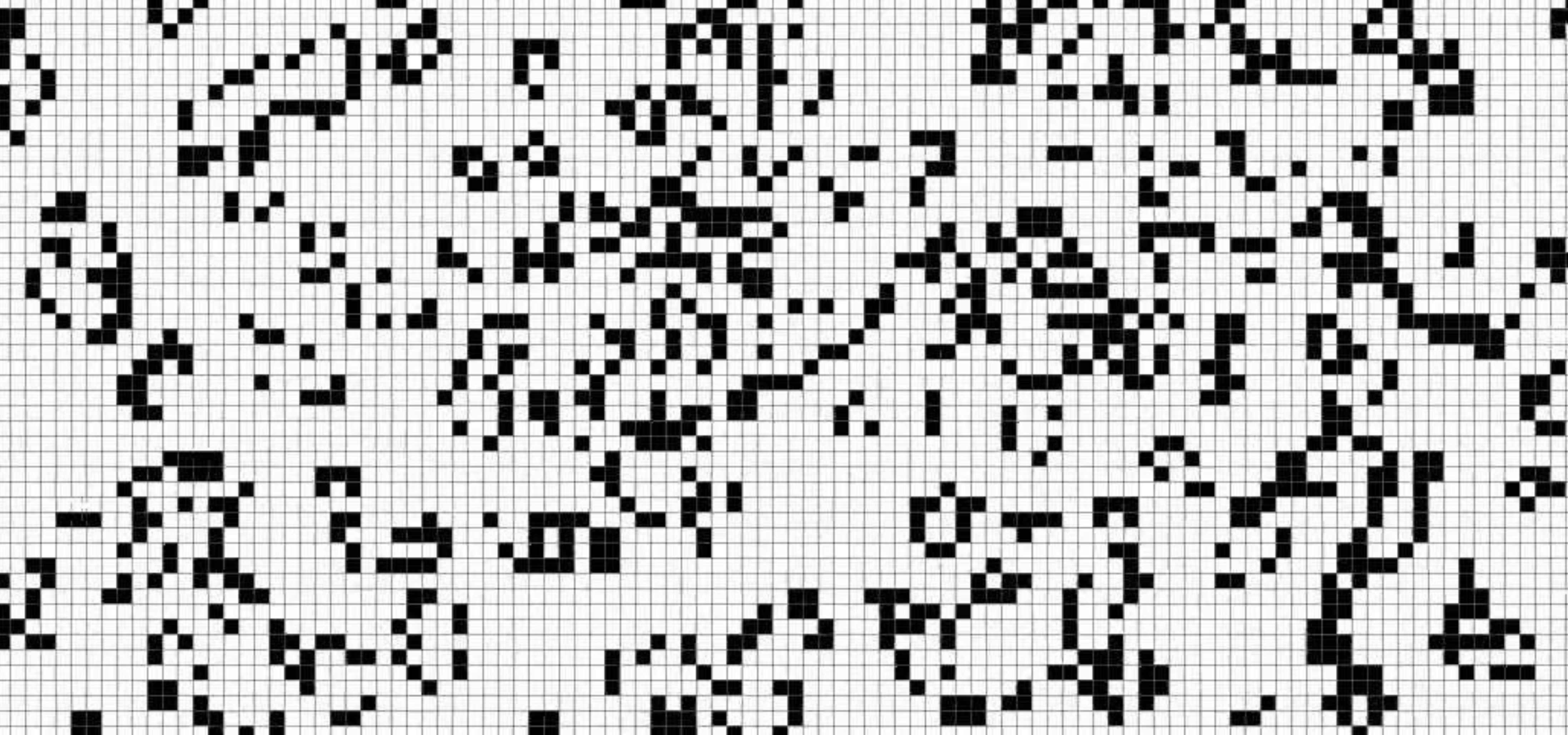
Can you tell whether a QCA is a circuit or not?

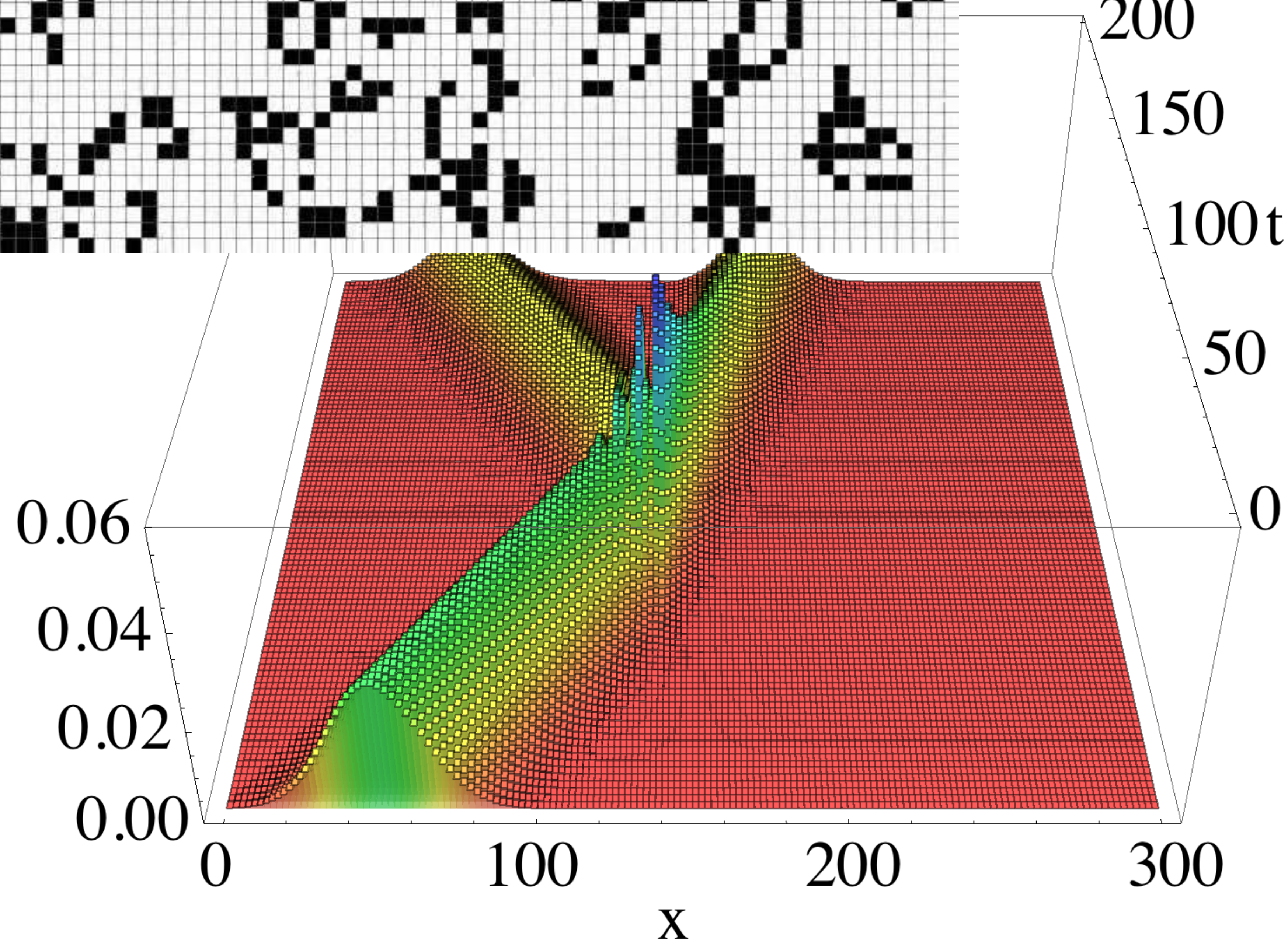
Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

What does it mean that α is a coarse graining of β?

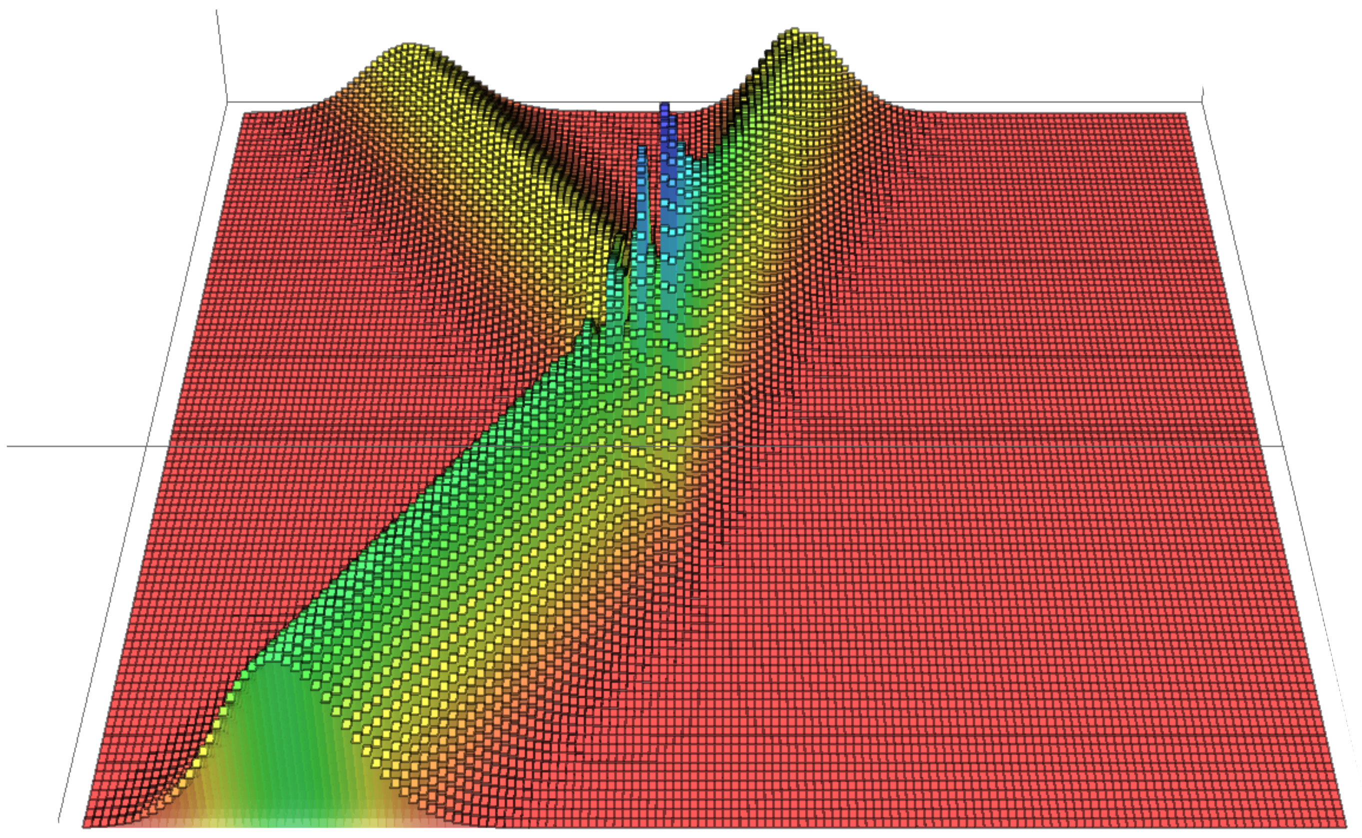
It means that the evolution of a **subset** of the degrees of freedom of α is described by β.

For 1D qubit QCA, the only QCA which admit a coarse graining are those **which do not propagate information**.





0.06
0.04
0.02
0.00
0
100
200
300
x
0
50
100
150
200
t



Summary

- QCA are the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems. A (possibly infinite) quantum circuit is a QCA but that is not always the case.

- We addressed two classification problem:
  1) Which **local rules** give rise to translation invariant QCAs?
     The problem is solved for qubit QCA with von Neumann neighborhood. These QCAs are all madeup of shifts, C-phase gates and local unitaries
  2) Which QCA are quantum circuits and which ones are not?
     The **index theory** (flow of information) solves the problem in dimension 1 (general case) and 2 (translation invariant case). A QCA is circuit iff its index is zero

- If the evolution of a subset of the degrees of freedom of QCA α is described by a QCA β, we say that β is a coarse graining of α.
  Not all QCA admit a coarse graining. For qubit QCA in one dimension, the only QCA which admits a coarse graining are the ones which do not propagate information

Summary

What is a QCA?

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell when a QCA is a circuit or not?

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell when a QCA is a circuit or not?

Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell when a QCA is a circuit or not?

Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

What does it mean that a QCA is a coarse graining of another QCA?

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell when a QCA is a circuit or not?

Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

What does it mean that a QCA is a coarse graining of another QCA?

What does it mean that a QCA is a coarse graining of another QCA?

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

Qubit QCAs with **von Neumann nieghborhood**, are all made up of layers of C-phase, local unitaries and shifts.

Can you tell when a QCA is a circuit or not?

Yes for dimension 1 (general) and 2 (translation invariance): a QCA is a circuit if and only if **the index** is zero. In higher dimensions we don't know.

What does it mean that a QCA is a coarse graining of another QCA?

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

For **qubit** QCAs with **von Neumann nieghborhood**,

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Summary

What is a QCA?

It the most general **reversible** and **local** discrete time evolutions of a lattice of quantum systems.

Which local rules give rise to a QCA?

QCA: formal definition

Se fosse finito dimensionale, vorrebbe dire coniugare
con una unitaria, ma con queste algbere infinite con

eisemberg picture: quantum
stem <—> algebra of observables

Quasi local algebra:

Local observables

automorphism (one to one map)

a QCA is local automorphism of A such that

Local, i.e.

example: circuit and shift

Classification of the update rule

he update rule may vary form site
site

if th rule is the same, the QCA is
translationally invariant

Which rule give rise to a QCA?

qubit von Neumann neighboring
scheme arbitrary dimension

Classification of the update rule

he update rule may vary form site
site

if th rule is the same, the QCA is
translationally invariant

Which rule give rise to a QCA?

qubit von Neumann neighboring
scheme arbitrary dimension

QCA vs quantum circuits

Classificare le regole è presto infattibile. c'è un altro problema di classificazione che sulla carta piu abbordabile come facciamo a sapere se un automa e' un circuito? COme e' realizzabile come circuito

Gli automi sono un gruppo

Qualunque automa con ancilla infite è un circuito quantistico

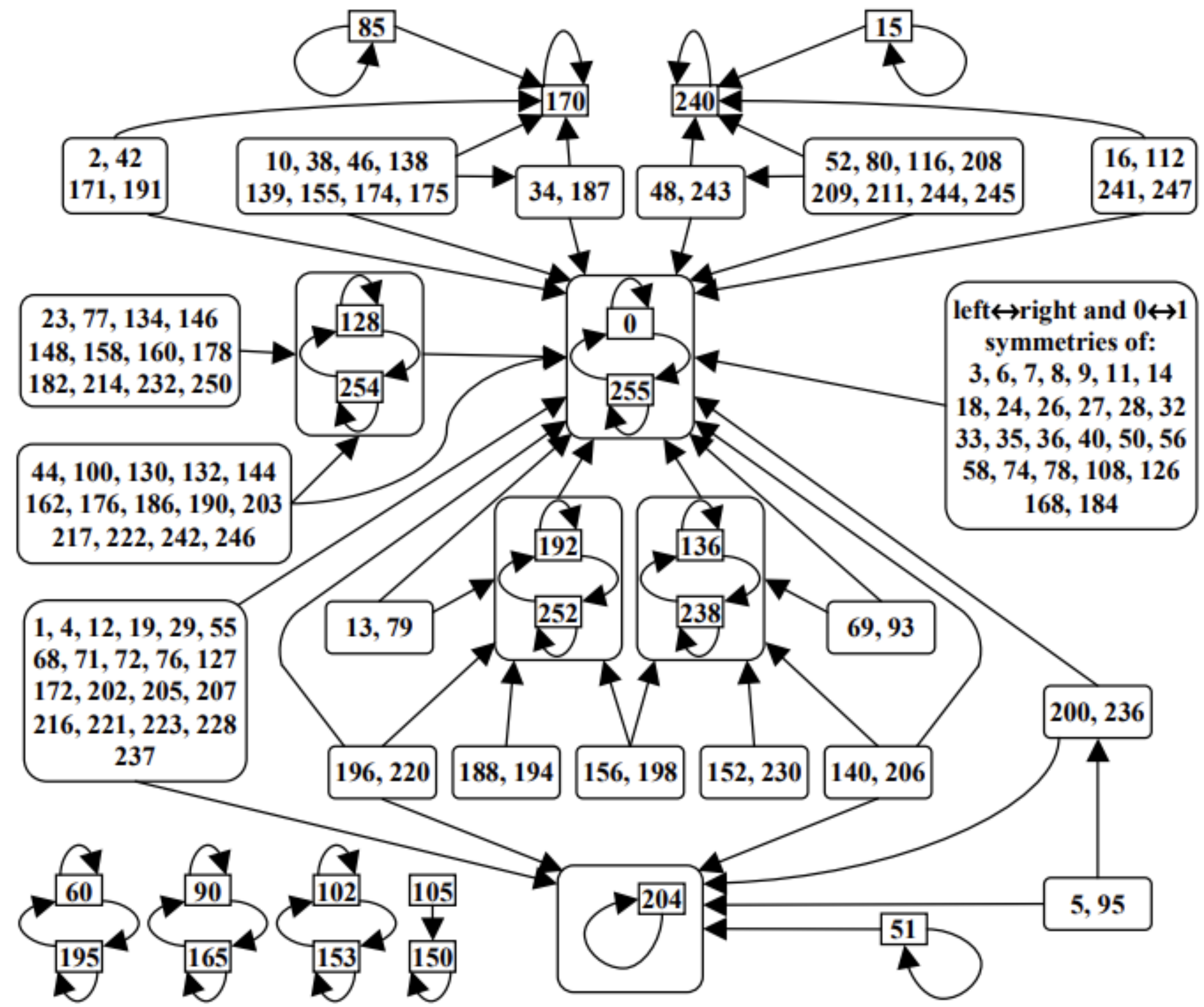
I circuiti sono un sottogruppo normale

Ha senso parlare del quoziente classi di equivalenza modulo circuiti

COme e' fatto questo gruppo?

Coarse graining in classical CA

Il coarse graining studiato da Israeli
e goldenfeld disegni



Complexity classification of C

Come si fa nel caso quantis

Come si fa nel caso quar

Coarse graining in QCA

Come lo definiamzo noi

disegni

Risultati

The index of a QCA

In 1 D c'e un flusso indice che ci dice questa cosa Q = Z

In dimensione pi`¨alta è tutto più complesso

Problema affine studiato da Hastings e Friedman in 2D. Noi risuciamo a dimostrarlo per il caso che ci interessa TI infinito

Summary

I have a quantum system which evolves with the Hamiltonian $H = H_0 + V$

Can I simulate a **scattering** experiment with a **discrete time** quantum simulator?

Yes, under certain conditions.

Summary

I have a quantum system which evolves with the Hamiltonian $H = H_0 + V$

Can I simulate a **scattering** experiment with a **discrete time** quantum simulator?

Yes, under certain conditions.

What is **error** of the simulation?

Summary

I have a quantum system which evolves with the Hamiltonian $H = H_0 + V$.

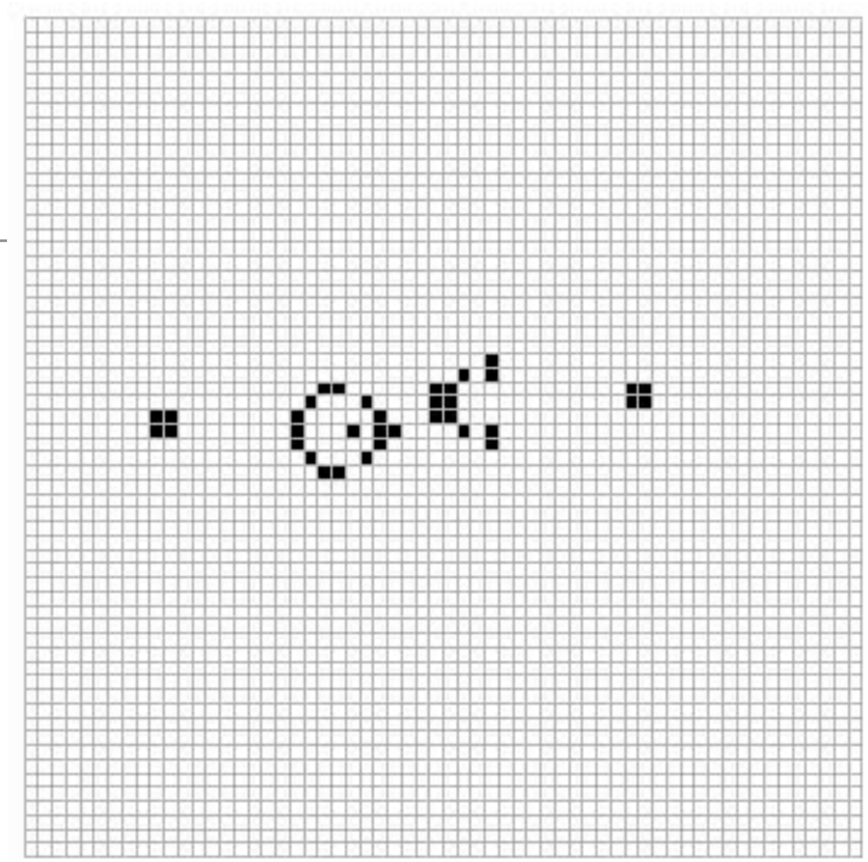
Can I simulate a **scattering** experiment with a **discrete time** quantum simulator?

Yes, under certain conditions.

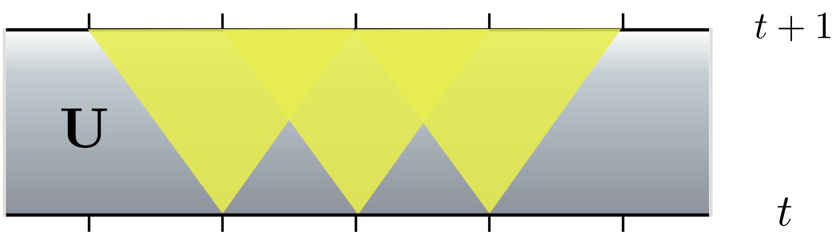
What is **error** of the simulation?

Linear in the time step and **quadratic** in V.

Cellular automata



A Cellular automaton is a lattice of systems with some local update rule (we have a “lightcone”)



Several application:

- computation
- simulation
- topological phases of matter