Derivation of Standard Quantum Theory via State Discrimination

Hayato Arai[1,(2)] and Masahito Hayashi[3,4,2]

RIKEN[1], Nagoya University[2],
The Chinese University of Hong Kong, Shenzhen[3], International Quantum Academy[4]

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Self-Introduction and Abstract

- Hayato Arai
  - Post-Doc at Riken, Group of Bartosz Regula
  - (Now) Mainly working on Non-IID Hypothesis Testing
  - Working on Foundation (in Ph.D Thesis)

Today's Talk

- Field : Foundation of Quantum Theory
- Aim : Derivation of Model of Quantum Theory
- Method : State Discrimination in General Probabilistic Theories
- Results (details in later)
  1. A Tight Bound for 2-State Discrimination in General Models
  2. Equivalent Condition for Violation of Quantum Bound
  3. **Derivation of Quantum Theory via State Discrimination!**

Motivation of GPTs

- **General Probabilistic Theories (GPTs)** is a modern general structure focusing on probabilistic structures obtained by states and measurements
- The requirement of GPTs is weak $\rightarrow$ there are many available models in GPTs except for quantum and classical theory.
- The aim of GPTs is to **find a "good" postulate to single out QT**
- Preceding studies are imperfect because some cannot characterize quantum theory uniquely, others are not operationally meaningful.

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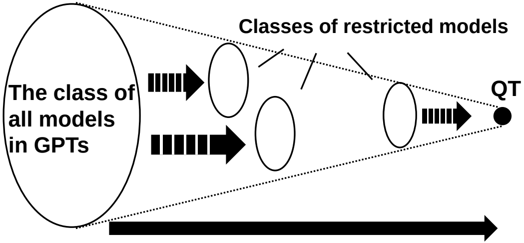

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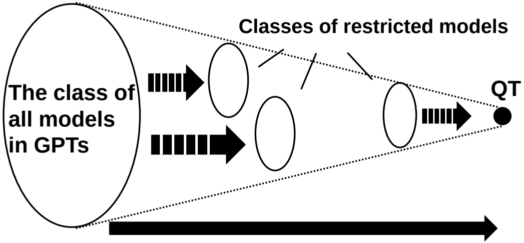

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Postulates	Operational Meaning	Single Out QT
No-Cloning [Barnum2006]	No-Go	$\times$
Tsirelson's bound [Barnum2010]	Bound Performance	$\times$
Purification $+\alpha$ [Chiribella2011]	?	$\checkmark$
Bit-symmetry $+\alpha$ [Barnum2019]	?	$\checkmark$
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2-State Discrimination [Today]	Bound Performance	$\checkmark$

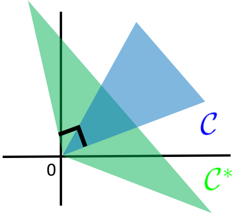
Preliminary

- positive (proper) cone $\mathcal{C} \subset \mathcal{V}$

$\mathcal{V}$: Finite-Dimensional Real-Vector Space
(with Inner Product $\langle\ ,\ \rangle$)
  - closed, convex, has non-empty interior
  - $\forall x \in \mathcal{C}$, $\forall r \geq 0$, $rx \in \mathcal{C}$
  - $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$
- dual cone $\mathcal{C}^* := \{f \in \mathcal{V}^* \mid f(x) \geq 0 \ \forall x \in \mathcal{C}\}$
- A Typical Example
  - $\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}) \quad (\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})^* := \{f(x) := \mathrm{Tr}\, xy \mid y \in \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})\} \simeq \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}))$

Assumption and Notation

$\mathcal{H}$: finite-dim. Hilbert sp.
$\mathcal{L}_{\mathrm{H}}(\mathcal{H})$: set of Hermitian matrices on $\mathcal{H}$
$\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})$: set of positive semi-definite matrices on $\mathcal{H}$



- In this study, we mainly consider the case $\mathcal{V} = \mathcal{L}_{\mathrm{H}}(\mathcal{H})$ with the Frobenius inner product $\langle X, Y \rangle = \mathrm{Tr}\, XY$

Definition of a model GPTs

A model of GPTs is defined by $\mathcal{C} \subset \mathcal{V}$
with an normalization effect $u \in \mathcal{V}^*$

- State ρ (Generalization of density matrix)
  - $\rho \in \mathcal{C}$
  - $u(\rho) = 1$
- Measurement $\{M_i\}_i$ (Generalization of POVM)
  - $M_i \in \mathcal{C}^*$

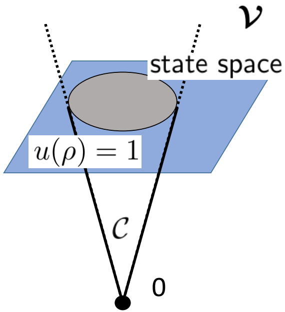
  $\Leftrightarrow$ $M_i(\rho) \geq 0 \ (\forall \rho \in \mathcal{C})$
  - $\sum_i M_i = u$
- When a state ρ is measured by $\{M_i\}$

  $\rightarrow$ The outcome i is obtained w.p. $p_i = M_i(\rho)$

$\rightarrow$ The model of GPTs is determine by $\mathcal{C}$

Correspondence

GPT		QT
$\mathcal{C}$	$\leftrightarrow$	$\mathcal{L}^+_{\mathrm{H}}(\mathcal{H})$



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Requirement

- $\{p_i\}$ is probability distribution
  - $M_i(\rho) \geq 0$
  - $\sum_i M_i(\rho) = 1$

(Roughly) Definition of a model GPTs

Quantum Theory (QT) on Hilbert Space $\mathcal{H}$

- State ρ (density matrix)
  - $\rho \in \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})$
  - $\mathrm{Tr}\,\rho = 1$
- Measurement $\{M_i\}_i$ (POVM)
  - $M_i \in \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})$

    $\Leftrightarrow$ $\mathrm{Tr}\,\rho M_i \geq 0$ $(\forall \rho \in \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}))$
  - $\sum_i M_i = I$
- When a state ρ is measured by $\{M_i\}$

  $\rightarrow$ The outcome i is obtained w.p. $p_i = \mathrm{Tr}\,\rho M_i$

$\rightarrow$ The model of QT is determine by $\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})$

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- $\{p_i\}$ is probability distribution
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Definition of a model GPTs

A Q-like model of GPTs is defined by $\mathcal{C} \subset \mathcal{L}_{\mathrm{H}}(\mathcal{H})$

- State ρ
  - $\rho \in \mathcal{C}$
  - $\mathrm{Tr}\,\rho = 1$
- Measurement $\{M_i\}_i$
  - $M_i \in \mathcal{C}^* \ (\subset \mathcal{L}_{\mathrm{H}}(\mathcal{H}))$

    $\Leftrightarrow$ $\mathrm{Tr}\,\rho M_i \geq 0 \ (\forall \rho \in \mathcal{C})$
  - $\sum_i M_i = I$
- Probability to get an outcome i
  - The outcome i is obtained w.p. $p_i = \mathrm{Tr}\,\rho M_i$

$\rightarrow$ A Q-like is determined by $\mathcal{C}$

Correspondence

GPT		QT
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Requirement

- $\{p_i\}$ is probability distribution
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Definition of a model GPTs

An example of Q-like models
defined by $\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_A) \otimes \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_B) \subset \mathcal{L}_{\mathrm{H}}(\mathcal{H}_A \otimes \mathcal{H}_B)$

- State ρ
  - $\rho \in \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_A) \otimes \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_B)$
  - $\mathrm{Tr}\,\rho = 1$
- Measurement $\{M_i\}_i$
  - $M_i \in \left(\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_A) \otimes \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_B)\right)^*$

    $\Leftrightarrow$ $\mathrm{Tr}\,\rho M_i \geq 0 \ (\forall \rho \in \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_A) \otimes \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_B))$
  - $\sum_i M_i = I$

  ex: Partial Transposed Entanlement (Beyond POVMs)
- Probability to get an outcome i
  - The outcome i is obtained w.p. $p_i = \mathrm{Tr}\,\rho M_i$

$\rightarrow$ A Q-like is determined by $\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_A) \otimes \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_B)$

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Isomorphism in models of GPTs

An isomorphism f from a model $\boldsymbol{G} = (\mathcal{C}, u)$ to $\tilde{\boldsymbol{G}} = (\tilde{\mathcal{C}}, \tilde{u})$ is defined as

- f is a linear isomorphism from $\mathcal{V}$ to $\tilde{\mathcal{V}}$ $(\dim(\mathcal{V}) = \dim(\tilde{\mathcal{V}}))$
- $\tilde{\mathcal{C}} = f(\mathcal{C})$
- $\tilde{u} \circ f = cu$ for a constant $c > 0$

- $\tilde{\rho} = \frac{1}{c} f(\rho)$
- $\tilde{M}_i = cM_i \circ f^{-1}$

$\rightarrow$ $\tilde{M}_i(\tilde{\rho}) = cM_i \circ f^{-1} \left(\frac{1}{c} f(\rho)\right) = M_i(\rho)$

- Therefore, the two model $\boldsymbol{G}$ and $\tilde{\boldsymbol{G}}$ are equivalent from the viewpoint of probabilistic structures obtained from states and measurements

- Any model with d^2-dimensional vector space can be isomophic to a Q-like model

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2-State Discrimination

- Given an unknown state ρ prepared as ρ_0, ρ_1 with probability $p, 1-p$, to identify the state ρ by one-shot measurement $\boldsymbol{M} = \{M_0, M_1\}$ with high probability
- Total error probability is given as

$$\mathrm{Err}(\rho_0; \rho_1; p; \boldsymbol{M}) := p \operatorname{Tr} \rho_0 M_1 + (1-p) \operatorname{Tr} \rho_1 M_0$$

- In QT, the error probability is bounded as Quantum Bound (Helstrom Bound)

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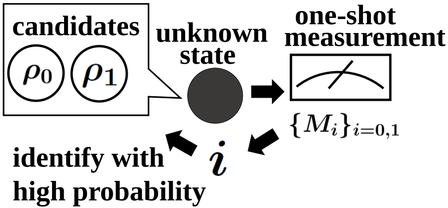

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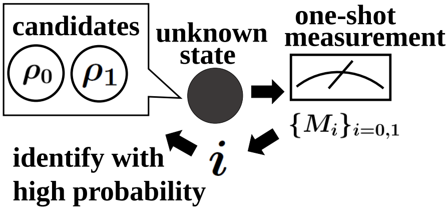

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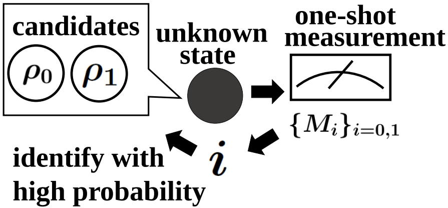

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- In the case $\mathcal{C} = \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_A) \otimes \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_B)$, there exists non-orthogonal perfectly distinguishable states ρ_0, ρ_1

$\rightarrow$ $\mathrm{Err}(\rho_0; \rho_1; p; \boldsymbol{M}) = 0$, and $\operatorname{Tr} \rho_0 \rho_1 > 0$ ($\Rightarrow \| \rho_0 - \rho_1 \| < 2$)

$\rightarrow$ Violates Quantum Bound ($p = 1/2$)

Example

$$M_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \Gamma\left(\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \right), \quad M_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The measurement $\{M_0, M_1\}$ can perfectly discriminate non-orthogonal separable states ρ_0, ρ_1 defined as

$$\rho_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \rho_1 := \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

- They satisfies $\mathrm{Tr}\, \rho_i M_j = \delta_{ij}$
- However, $\mathrm{Tr}\, \rho_0 \rho_1 > 0$

General Bound of Error Probability in 2-State Discrimination

- For a two-outcome measurement $\boldsymbol{M} = \{M_0, M_1\}$, we define

$$r(\boldsymbol{M}) := \lambda_{\max}(M_i) - \lambda_{\min}(M_i), \quad r'(\boldsymbol{M}, i) := \lambda_{\max}(M_i) + \lambda_{\min}(M_i).$$

- Because $M_0 + M_1 = I$, the value $r(\boldsymbol{M})$ is independent of i
  $\because$ $\lambda_{\max}(M_1) = 1 - \lambda_{\min}(M_0)$, $\lambda_{\min}(M_1) = 1 - \lambda_{\max}(M_0)$

Theorem 1 (General Lower Bound)

Consider a (Q-like) model. Any pair of two states ρ_0, ρ_1 and any measurement $\boldsymbol{M} = \{M_0, M_1\}$ in the model satisfy

$$\mathrm{Err}(\rho_0; \rho_1; p; \boldsymbol{M}) \geq \frac{1}{2} - \frac{1}{2}\|p\rho_0 - (1-p)\rho_1\|_1 r(\boldsymbol{M}) - \frac{1}{2}(2p-1)\left(r'(\boldsymbol{M}, 0) - 1\right). \quad (1)$$

- $\boldsymbol{M}$ is a POVM $\Rightarrow r(\boldsymbol{M}) \leq 1$, $\boldsymbol{M}$ is optimal in QT $\Rightarrow r'(\boldsymbol{M}, i) = 1$

$\rightarrow$ (1) reproduce Quantum Bound.

- There exist ρ_i and p satisfies the equality of (1) (Tight!)

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- In the case $p = 1/2$, the tight bound is simply given by $r(\boldsymbol{M})$
- By applying this, an equivalnt condition for violation of Quantum Bound

Equivalent Condition for Violation of Quantum Bound

Theorem 2 (Equivalent Condition for Violation of Quantum Bound)

Consider a (Q-like) model. Given a measurement $\boldsymbol{M} = \{M_0, M_1\}$ in the model, the following two conditions are equivalent:

1. There exist two states ρ_0 and ρ_1 in the model such that
$$\mathrm{Err}(\rho_0; \rho_1; p = \frac{1}{2}; \boldsymbol{M}) < \frac{1}{2} - \frac{1}{2} \| \frac{1}{2}\rho_0 - \frac{1}{2}\rho_1 \|_1. \tag{3}$$
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- If there exists a measurement $\boldsymbol{M}$ with $r(\boldsymbol{M}) > 1$, the model violates Quantum Bound
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Q. Does this uniquely characterize QT? → Yes!

Equivalent Condition for Violation of Quantum Bound

Theorem 2 (Equivalent Condition for Violation of Quantum Bound)

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Derivation of Quantum Theory via State Discrimination

Theorem 3 (Derivation of QT via state discrimination)

Consider a (general) model $\boldsymbol{G}$. The following conditions are equivalent:

1. $\boldsymbol{G} = \boldsymbol{QT}$
2. There exists an isometric map $\boldsymbol{G}$ to (Q-like) model $\tilde{\boldsymbol{G}}$ such that
   - A. Any state in $\tilde{\boldsymbol{G}}$ is a density matrix (quantum state)
   - B. Any state ρ_0, ρ_1, $0 < p < 1$, and any measurement $\boldsymbol{M}$ in $\tilde{\boldsymbol{G}}$ satisfies Quantum Bound

- Condition 2
  $\Leftrightarrow$ embedding state space into quantum state space with satisfying Quantum Bound
- No model satisfies Quantum Bound but violates other properties of quantum theory
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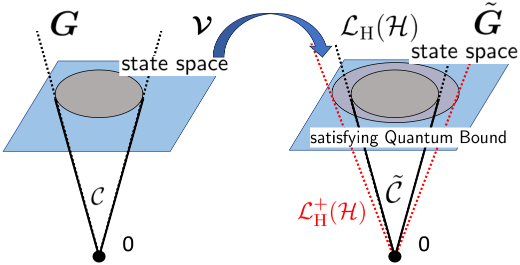

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REMARK (for experts)

- isometric map exists $\Leftrightarrow$ background dimensions are the same
- if background dimension is not square number, both conditions 1 and 2 are false
- if background dimension is a square number, both conditions 1 and 2 are true

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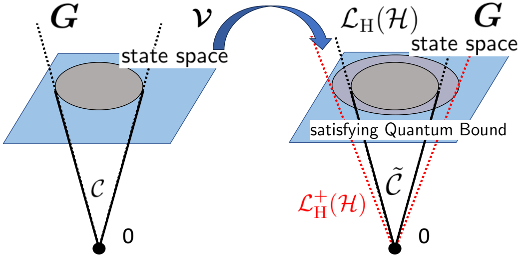

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- Due to the condition $\tilde{\mathcal{C}} \subsetneq \mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H})$, there exists a measurement beyond POVMs
- Even if a measurement is beyond POVMs,
  it is not trivial that there exists a measurement $\boldsymbol{M}$ with $r(\boldsymbol{M}) > 1$
- Theorem 3 is non-trivial

Why Classical Theory Violates Condition 2?

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- For existence of isomorphism to a Q-like model, the dimension of classical theory is d^2

→ There exists d^2-number of perfectly distinguishable classical states

→ They are embeded into density matrices

→ They must be non-orthogonal

→ A pair of non-orthogonal perfectly distinguishable states violates Quantum Bound in $p = 1/2$

$$\mathrm{Err}(\rho_0; \rho_1; p = \frac{1}{2}; \boldsymbol{M}) \not\geq \frac{1}{2} - \frac{1}{2}\|\frac{1}{2}\rho_0 - \frac{1}{2}\rho_1\|_1. \tag{4}$$

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Summary and Take-Home Message

- There are many models in GPTs except for quantum and classical theory
- There was no operational meaningful postulate to single out QT
- We show that **QT is characterized by Quantum Bound** of error probability in 2-state discrimination
- **No model satisfies Quantum Bound** but **violates other properties** of quantum theory

- New postulate to derive QT through performances for information tasks!
- Performance for state discrimination characterize the performances for all other tasks!

Open Problems

1. Extension to the hypothesis testing and n-shot asymptotic setting
   - Instead of the sum of errors, can we deal with each types of errors $\mathrm{Tr}\,\rho_0 M_1$ and $\mathrm{Tr}\,\rho_1 M_0$
   - What is the general asymptotic rate?
2. Relaxation of the condition about the existence of isomophism
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   - We need more general measure of the performance for state discrimination

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