Optimal convex approximation of quantum superposition

Quantum TUT workshop 2024

NTT Communication Science Laboratories

Seiseki Akibue, Seiichiro Tani

NICT Quantum ICT Laboratory

Go Kato

arXiv: 2301.06307
npj quantum info. 10, 3 (2024)

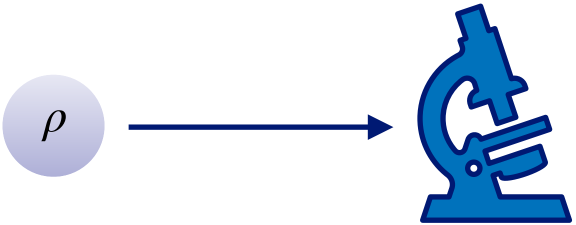
Background

- Quantum superposition enables us to access huge computational space. (N-qubit pure state: $\alpha_{0\cdots 0}|0\cdots 0\rangle + \alpha_{0\cdots 1}|0\cdots 1\rangle + \cdots \alpha_{1\cdots 1}|1\cdots 1\rangle \simeq \mathbb{C}^{2^N}$)

- Superposition $\neq$ convex combination (=probabilistic mixture)
  - Pure state is not a convex combination of distinct states.
  - Unitary trans. is not a convex combination of distinct processes.

Background

- Superposition $\neq$ convex combination (=probabilistic mixture)
  - Pure state is not a convex combination of distinct states.
  - Unitary trans. is not a convex combination of distinct processes.
- However, the difference cannot be perfectly distinguished with a finite number of copies.



$$\rho = |+\rangle\langle+|$$

or

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

Background

- This raises the question, "Can we improve an approximation of a target pure state by using a convex combination of available states?"

Target pure state
(Superposition of available states)

$$|\phi\rangle = \alpha_0 |\psi_0\rangle + \alpha_1 |\psi_1\rangle + \alpha_2 |\psi_2\rangle \qquad \simeq$$

Available states

ψ_0

ψ_1

ψ_2

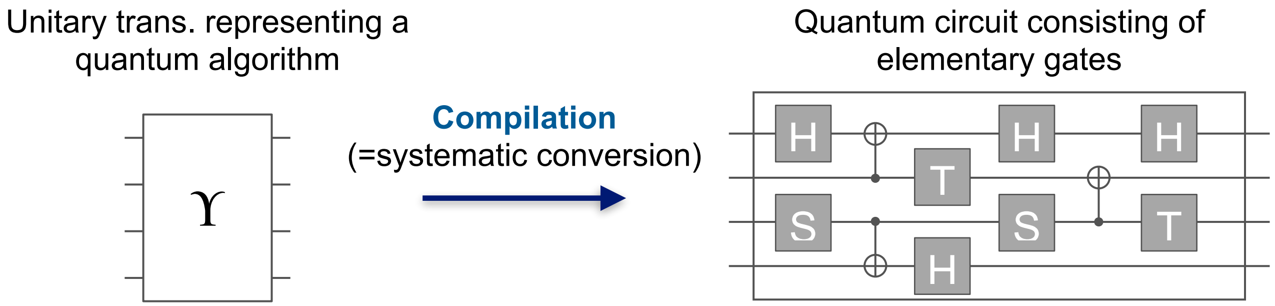
Convex combination

$$\sum_{x=0}^{2} p(x)\psi_x$$

Improve approximation?

Background|compilation

- In FTQC, we have noiseless gate operations, called elementary gates. They vary depending on the encoding of the logical qubits, e.g., Clifford+T for the surface code.
- It is important to **systematically** convert a given unitary trans. into a quantum circuit consisting of elementary gates.



- Example: compilation by using one kind of elementary gate.

Target unitary trans.

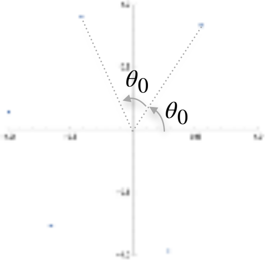
$$R_z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

Compilation →

Quantum circuit consisting one kind of elementary gate

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_0} \end{pmatrix}$$

Implementable angles θ with fewer than 5 elementary gates (blue dots)



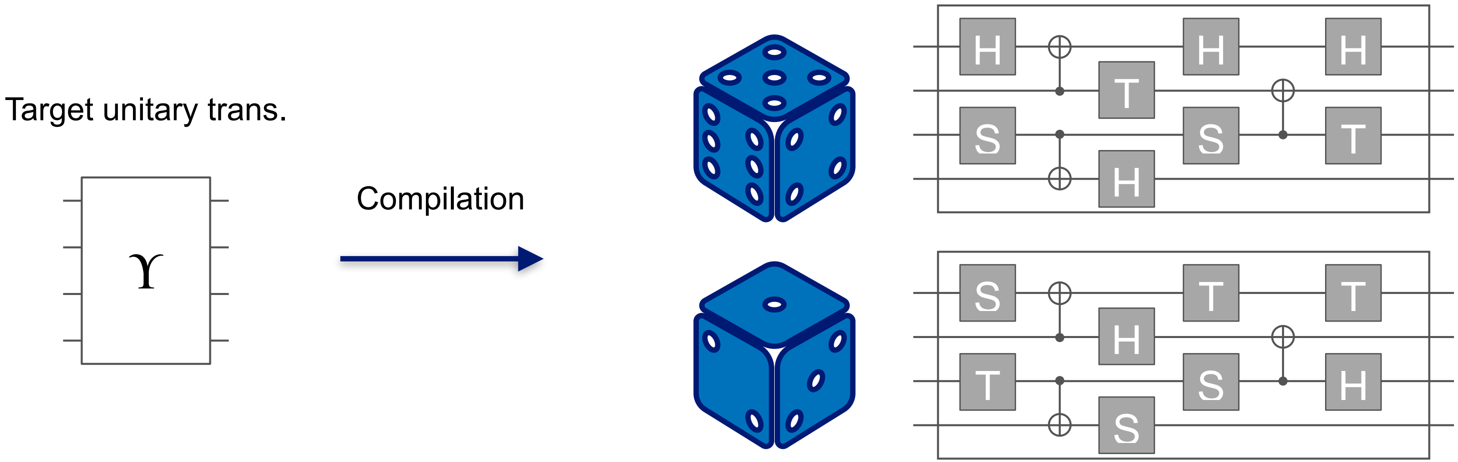
The set of implementable unitary trans. forms a finer ϵ-net if one enlarges the circuit size.

Implementable angles θ with fewer than 300 elementary gates (blue dots)

Final goal until 2016: Find the smallest circuit to implement the target unitary trans. within a desired approximation error.

Background|compilation

- In 2016, Campbell and Hastings have independently found that the approximation error can be reduced by probabilistically sampling unitary trans.

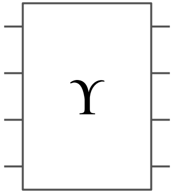


Background|compilation

NTT

- Approximating a target unitary trans. by using a convex combination is better.

Target unitary trans.



$\simeq$

Available unitary trans.
(e.g. unitary trans. implementable by $N(\leq 300)$ gates.)

Υ_0

Υ_1

Υ_2

Convex combination

$$\sum_{x=0}^{2} p(x)\Upsilon_x$$

Improve approximation!

Summary of our contributions

Open problem: What is the fundamental limitation of the unitary (state) approximation by using their convex combinations?

Difficulty: How can we optimize $p(x)$ in the convex combination? Only a few optimal solutions have been known since the high-dimensional geometry of unitary and state is complicated.

In this talk, we show

(1) **tight bounds on the reduction rate of the approximation error** by using optimal $p(x)$,

(2) a construction of **an efficient algorithm to optimize** $p(x)$,

(3) several numerical demonstrations,

(4) Other applications of our method for **analyzing resource measure**.

Convex approximation of unitary

NTT

We have derived the tight inequalities on the reduction rate of the approximation error

Target unitary trans.

Available unitary trans.

Theorem 1. For any unitary Υ and set of unitaries $\{\Upsilon_x\}_x$ acting on $\mathbb{C}^d$, it holds that

$$\frac{4\delta_\Upsilon}{d}\left(1-\frac{\delta_\Upsilon}{d}\right) \le \min_p \frac{1}{2}\left\|\Upsilon - \sum_x p(x)\Upsilon_x\right\|_\diamond \le \epsilon^2$$

with

$$\epsilon_\Upsilon = \min_x \frac{1}{2}\left\|\Upsilon - \Upsilon_x\right\|_\diamond$$

$$\delta_\Upsilon = 1 - \sqrt{1-\epsilon_\Upsilon^2}$$

$$\epsilon = \max_\Upsilon \epsilon_\Upsilon$$

$$\simeq \frac{2\epsilon_\Upsilon^2}{d}$$

The approximation error by using the optimal convex combination

The **worst-case** approximation error by using a single available unitary trans.

Convex approximation of unitary

NTT

As a corollary of Theorem 1, we obtain

Corollary 1. For any set of unitaries $\{\Upsilon_x\}_x$ acting on $\mathbb{C}^2$, it holds that

$$\max_{\Upsilon} \min_{p} \frac{1}{2} \left\| \Upsilon - \sum_x p(x) \Upsilon_x \right\|_{\diamond} = \left(\max_{\Upsilon} \min_{x} \frac{1}{2} \left\| \Upsilon - \Upsilon_x \right\|_{\diamond} \right)^2$$

Convex approximation of unitary

As a corollary of Theorem 1, we obtain

Corollary 1. For any set of unitaries $\{\Upsilon_x\}_x$ acting on $\mathbb{C}^2$, it holds that

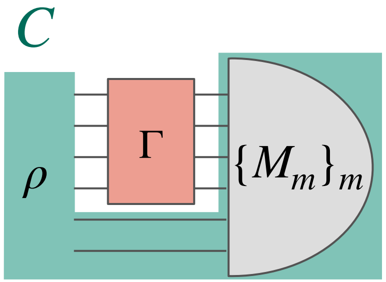
$$\max_{\Upsilon} \min_{p} \frac{1}{2} \left\| \Upsilon - \sum_x p(x) \Upsilon_x \right\|_{\diamond} = \left(\max_{\Upsilon} \min_{x} \frac{1}{2} \left\| \Upsilon - \Upsilon_x \right\|_{\diamond} \right)^2$$

⟺ The optimal convex approximation reduces the worst-case approximation error **quadratically**.

⟺ When $\{\Upsilon_x\}_x$ forms an ϵ-net of unitary transformations, its convex hull forms an ϵ^2-net of unitary transformations.

Why is convex combination useful?

Let us define the probability distribution by a CPTP map Γ in a circuit C as

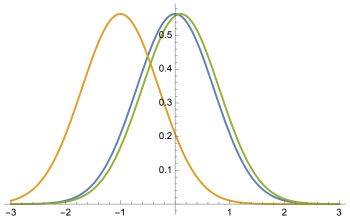


$$q(C,\Gamma) = \begin{pmatrix} tr[M_1(\Gamma \otimes id)(\rho)] \\ tr[M_2(\Gamma \otimes id)(\rho)] \\ \cdots \end{pmatrix}.$$

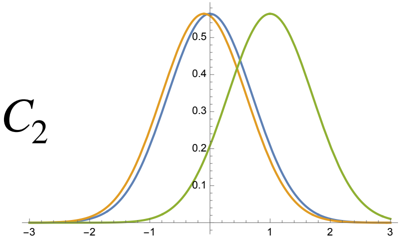
Then, we obtain

$$\frac{1}{2}\left|\left|\Upsilon - \Upsilon_x\right|\right|_\diamond = \max_C \frac{1}{2}\left|\left|q(C,\Upsilon) - q\left(C,\Upsilon_x\right)\right|\right|_1 \qquad \frac{1}{2}\left|\left|\Upsilon - \sum_x p(x)\Upsilon_x\right|\right|_\diamond = \max_C \frac{1}{2}\left|\left|q(C,\Upsilon) - \sum_x p(x) q\left(C,\Upsilon_x\right)\right|\right|_1$$

$C = C_1$



$C = C_2$



target distribution $q(C,\Upsilon)$
distribution $q(C,\Upsilon_1)$
distribution $q(C,\Upsilon_2)$

$p(x)$ is designed such that $q(C,\Upsilon_x)$ cancels out each other in $\sum_x p(x)q(C,\Upsilon_x)$ for any C.

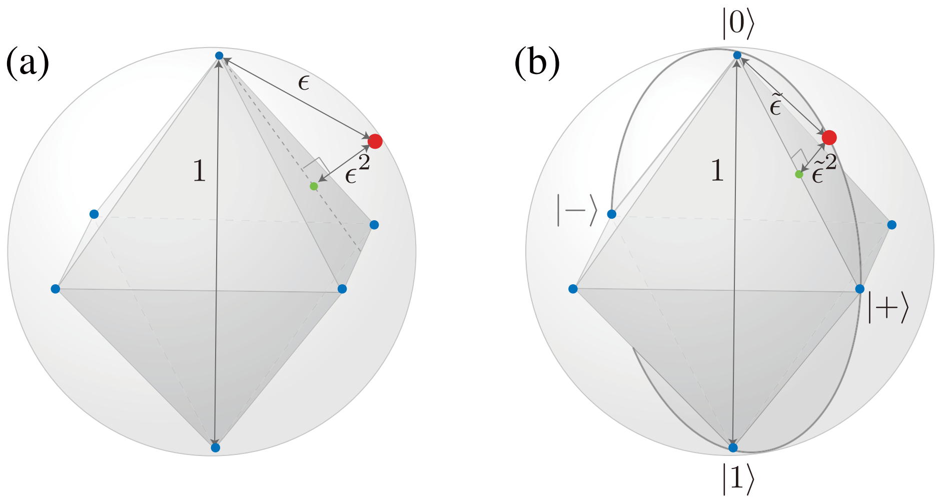
Convex approximation of state

Quadratic reduction of the approximation error when we synthesize a target qubit state by using the six eigenstates of Pauli ops. can be verified by using this theorem with

(a) $G = \{\mathbb{I}\}, S_G = \{\phi : |\phi\rangle \in \mathbb{C}^2\}$

(b) $G = \{\mathbb{I}, \theta\}, S_G = \{\phi : |\phi\rangle = \cos t|0\rangle + \sin t|1\rangle\}$

Complex conjugation



Theorem 2. Let $S_G := \{\phi : \forall U \in G, [U, \phi] = 0\}$ be the set of pure states invariant under a subgroup G of unitary and antiunitary operators. If $\{\phi_x\}_x$ is G-invariant,

$$\max_{\phi \in S_G} \min_{p} \frac{1}{2}||\phi - \sum_x p(x)\phi_x||_1 = \max_{\phi \in S_G} \min_{x} \left(\frac{1}{2}||\phi - \phi_x||_1\right)^2$$

Worst approximation error caused by probabilistic synthesis

Worst approximation error caused by deterministic synthesis

How can we obtain optimal $p(x)$?

We have to solve minimax optimization: $\min_p \frac{1}{2} \left\| \phi - \sum_x p(x)\phi_x \right\|_1 = \min_p \max_{0 \le M \le \mathbb{1}} tr\left[M \left(\phi - \sum_x p(x)\phi_x \right) \right]$.

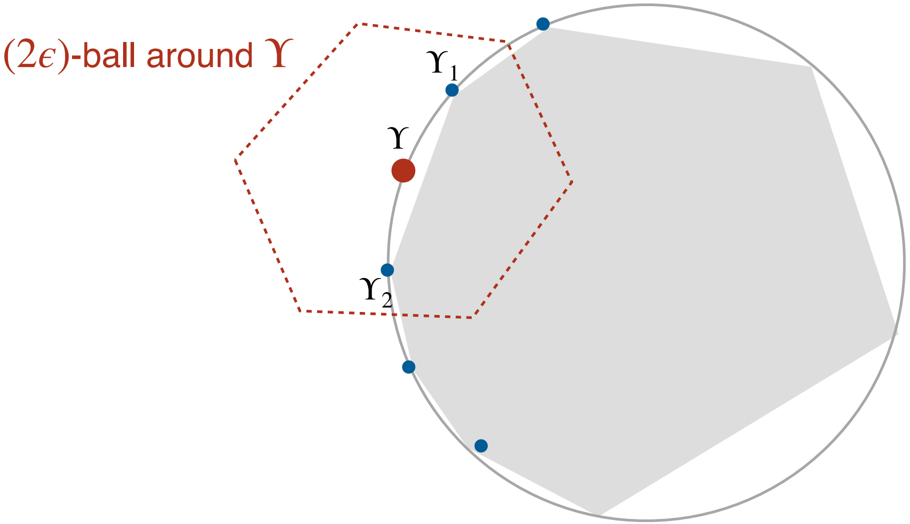
This is analytically intractable from previous studies. In contrast, we show that this can be algorithmically solved by an **SDP**.

However, we still need ϵ-net $\{\phi_x\}_{x \in X}$ consisting of available states to achieve the guaranteed quadratic reduction of the approximation error.

- In the context of compilation, we can obtain the ϵ-net by using a conventional compiler.
- However, $|X| = \Omega\left(\frac{1}{\epsilon}\right)$ is too large for efficient compilation.

How can we obtain optimal $p(x)$?

By exploiting a spherical representation of single-qubit unitary transformations, we obtain the following lemma.



Lemma 1. Optimal $p(x)$ that can be obtained by mixing an ϵ-net S is attainable by mixing the intersection of S and the (2ϵ)-ball around Υ.

The size of the intersection is a **constant** independent from ϵ.

Construction of a probabilistic compiler

Efficient probabilistic synthesis algorithm for single qubit unitary trans.

INPUT: target unitary Υ, approximation error ϵ

OUTPUT: gate sequence realizing Υ_x according to $p(x)$

Theorem 3. There exists a probabilistic state synthesis algorithm that calls a deterministic state synthesis algorithm constant times such that

Efficiency: runtime is $polylog\left(\frac{1}{\epsilon}\right)$

Quadratic improvement: the approximation error achieved by this algorithm satisfies $\frac{1}{2}\left|\left|\Upsilon - \sum_x p(x)\Upsilon_x\right|\right|_\diamond \leq \epsilon^2$ while $\min_x \frac{1}{2}\left|\left|\Upsilon - \Upsilon_x\right|\right|_\diamond \leq \epsilon$.

In the algorithm, we call a conventional compiler with approximation error ϵ. Then, we can achieve approximation error ϵ^2 by probabilistic compilation

Numerical demonstrations

NTT

Remaining problems:

- Can we achieve (more than if $d > 2$) quadratic reduction compared to ϵ_Υ for randomly sampled Υ?
- Does Lemma 1 hold for $d > 2$?

Theorem 1. For any unitary Υ and set of unitaries $\{\Upsilon_x\}_x$ acting on $\mathbb{C}^d$, it holds that

$$\frac{4\delta_\Upsilon}{d}\left(1-\frac{\delta_\Upsilon}{d}\right) \leq \min_p \frac{1}{2}\left|\left|\Upsilon - \sum_x p(x)\Upsilon_x\right|\right|_\diamond \leq \epsilon^2 \quad \text{with} \quad \begin{aligned} \epsilon_\Upsilon &= \min_x \frac{1}{2}\left|\left|\Upsilon - \Upsilon_x\right|\right|_\diamond \\ \delta_\Upsilon &= 1 - \sqrt{1-\epsilon_\Upsilon^2} \\ \epsilon &= \max_\Upsilon \epsilon_\Upsilon \end{aligned}$$

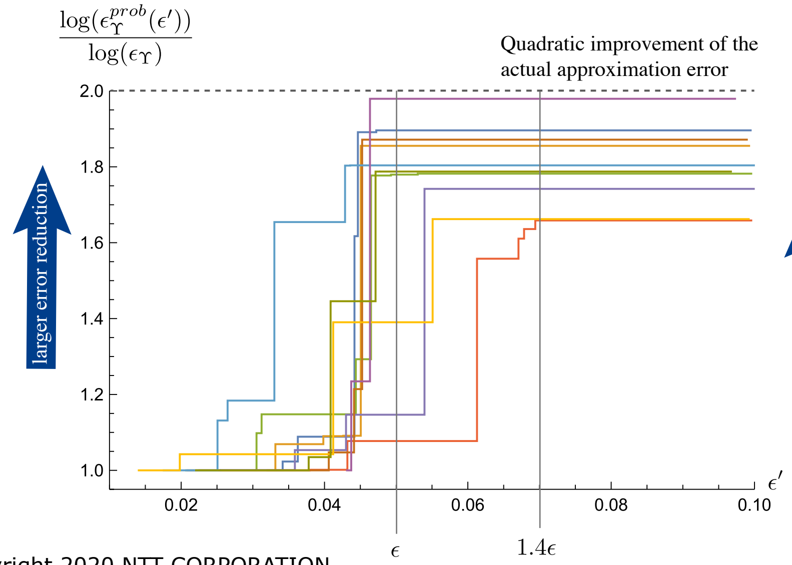
$$\simeq \frac{2\epsilon_\Upsilon^2}{d}$$

Numerical demonstrations

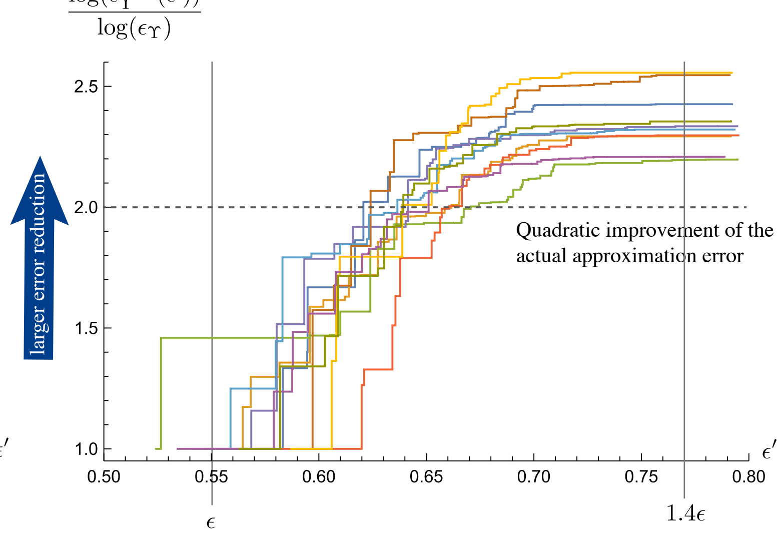
Numerics supports all the remaining problems.

$$\epsilon_{\Upsilon}^{prob}(\epsilon') := \min_{p} \frac{1}{2} \left\| \Upsilon - \sum_{x \in X(\epsilon')} p(x) \Upsilon_x \right\|_{\diamond} \qquad X(\epsilon') = \left\{ x : \frac{1}{2} \left\| \Upsilon - \Upsilon_x \right\|_{\diamond} \leq \epsilon' \right\}$$

Available unitary operators
= 10^5 randomly sampled U(2)

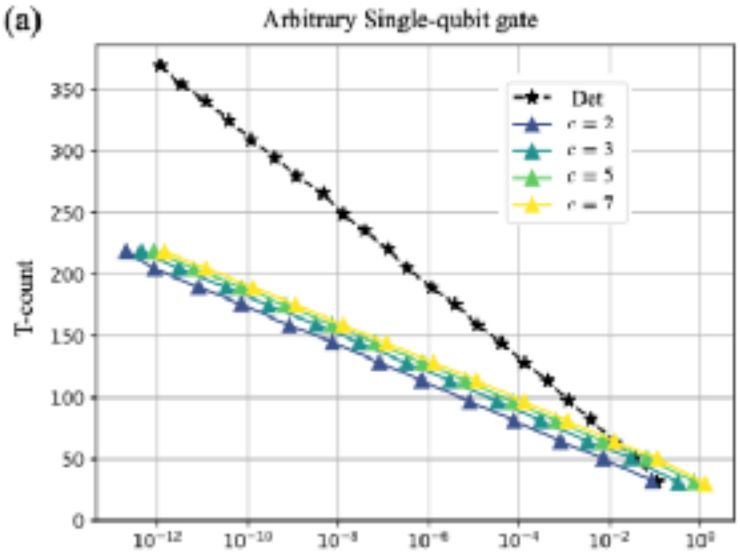


Available unitary operators
= 10^7 randomly sampled U(4)



Numerical demonstrations

Halve the T-count for compiling randomly sampled single-qubit unitary operations.



Approximation error ϵ

joint work with N. Yoshioka, Y. Suzuki, S. Endo, and Y. Tokunaga

Conventional (=deterministic) compilation

$$\#T \simeq 9\log_2\left(\frac{1}{\epsilon}\right)$$

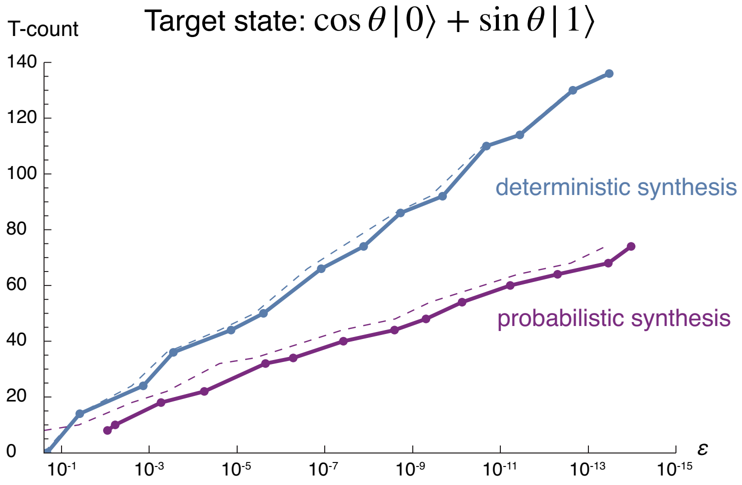
[RS compiler]

Probabilistic compilation

$$\#T \simeq 9\log_2\left(\frac{1}{\sqrt{\epsilon}}\right) = 4.5\log_2\left(\frac{1}{\epsilon}\right)$$

Numerical demonstrations

Halve the T-count for generating randomly sampled pure states.



Conventional (=deterministic) compilation

$$\#T \simeq 3\log_2\left(\frac{1}{\epsilon}\right)$$

[RS compiler]

Probabilistic compilation

$$\#T \simeq 3\log_2\left(\frac{1}{\sqrt{\epsilon}}\right) = 1.5\log_2\left(\frac{1}{\epsilon}\right)$$

Applications | Entanglement measure

NTT

We show an general lemma about the optimal convex approximation of states to obtain Theorem 2.

M. F. Sacchi, PRA 96, 042325 (2017)

Theorem 2. Let $S_G := \{\phi : \forall U \in G, [U, \phi] = 0\}$ be the set of pure states invariant under a subgroup G of unitary and antiunitary operators. If $\{\phi_x\}_x$ is G-invariant,

$$\max_{\phi \in S_G} \min_{p} \frac{1}{2} ||\phi - \sum_x p(x)\phi_x||_1 = \max_{\phi \in S_G} \min_{x} \left(\frac{1}{2} ||\phi - \phi_x||_1 \right)^2$$

Our general lemma contributes to the original motivation of the optimal convex approximation, quantifying quantum entanglement.

Applications | Entanglement measure

Exact formulas about entanglement measure w.r.t. the trace norm

Proposition [conjectured in A. Girardin et al., PRR 4, 023238 (2022)]
The trace norm between the Werner state ρ_q^{WER} (the isotropic state ρ_q^{ISO}) and SEP is given by

$$\min_{\sigma\in\mathbf{SEP}} ||\rho_q^{WER} - \sigma||_1 = 2q-1, \quad \min_{\sigma\in\mathbf{SEP}} ||\rho_q^{ISO} - \sigma||_1 = 2\frac{d^2-1}{d^2}\left(q - \frac{1}{d+1}\right)$$

This conjecture is proven by using a lemma for proving Thm.2

with $G = \{U \otimes U\}$ $(G = \{U \otimes U^*\})$ and $\{\phi_x\}_x = \{\phi \otimes \psi\}$.

Theorem 2. Let $S_G := \{\phi : \forall U \in G, [U, \phi] = 0\}$ be the set of pure states invariant under a subgroup G of unitary and antiunitary operators. If $\{\phi_x\}_x$ is G-invariant,

$$\max_{\phi\in S_G} \min_{p} \frac{1}{2}||\phi - \sum_x p(x)\phi_x||_1 = \max_{\phi\in S_G} \min_{x} \left(\frac{1}{2}||\phi - \phi_x||_1\right)^2$$

Applications | Entanglement measure

Exact formulas about entanglement measure w.r.t. the trace norm

Proposition [conjectured in A. Girardin et al., PRR 4, 023238 (2022)]
The trace norm between the Werner state ρ_q^{WER} (the isotropic state ρ_q^{ISO}) and SEP is given by

$$\min_{\sigma \in \mathbf{SEP}} ||\rho_q^{WER} - \sigma||_1 = 2q - 1, \quad \min_{\sigma \in \mathbf{SEP}} ||\rho_q^{ISO} - \sigma||_1 = 2\frac{d^2-1}{d^2}\left(q - \frac{1}{d+1}\right)$$

Note that we can compute these values by observing the closest separable state σ is also a Werner or isotropic state.

However, our technique does NOT need to find the closest separable state. Moreover, **it includes proof** for the region of q where a Werner or isotropic state is separable.

Thus, our technique has an advantage when **the closest separable state is unknown.**

Applications | Entanglement measure

Alternate succinct proof for the coincidence between entanglement and coherence

Proposition [J. Chen et al., PRA 94, 042313 (2016)]
Entanglement and coherence measures w.r.t. the trace norm about pure states coincide, i.e.,

$$\min_{\sigma \in \mathbf{SEP}} ||\Phi - \sigma||_1 = \min_{\rho \in I} ||\phi - \rho||_1$$

where

$$\begin{cases} |\Phi\rangle = \sum_i \alpha_i |ii\rangle, |\phi\rangle = \sum_i \alpha_i |i\rangle \\ I = conv(|i\rangle\langle i|) \end{cases}$$

This is also proven by using a lemma for proving Thm.2.

Theorem 2. Let $S_G := \{\phi : \forall U \in G, [U, \phi] = 0\}$ be the set of pure states invariant under a subgroup G of unitary and antiunitary operators. If $\{\phi_x\}_x$ is G-invariant,

$$\max_{\phi \in S_G} \min_p \frac{1}{2}||\phi - \sum_x p(x)\phi_x||_1 = \max_{\phi \in S_G} \min_x \left(\frac{1}{2}||\phi - \phi_x||_1\right)^2$$

Conclusion & Open problem

- We provide the fundamental limitations of convex approximation of unitary trans. and pure states.
- We construct an efficient algorithm to find the optimal probabilistic mixture.
- Our algorithm is compatible with many conventional deterministic compilers. It is sufficient to call a deterministic compiler constant times to achieve the optimal probabilistic compilation.
- The reduction rate of the circuit size depends on which deterministic compiler we call as a subroutine in our probabilistic compiler.

  (For the RS compiler, ~**50%** reduction is possible. For the SK compiler, ~**85%** reduction is possible.)
- Combination with a compiler used in Hamiltonian simulation such as Suzuki-Trotter decomposition would be promising.